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Physics Reports 371 (2002) 461–580

PHYSICS REPORTS

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Chaos, fractional kinetics, and anomalous transport

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Received 1 July 2002
editor: D.K. Campbell

Abstract

Chaotic dynamics can be considered as a physical phenomenon that bridges the regular evolution of systems with the random one. These two alternative states of physical processes are, typically, described by the corresponding alternative methods: quasiperiodic or other regular functions in the first case, and kinetic or other probabilistic equations in the second case. What kind of kinetics should be for chaotic dynamics that is intermediate between completely regular (integrable) and completely random (noisy) cases? What features of the dynamics and in what way should they be represented in the kinetics of chaos? These are the subjects of this paper, where the new concept of fractional kinetics is reviewed for systems with Hamiltonian chaos. Particularly, we show how the notions of dynamical quasi-traps, Poincaré recurrences, Lévy flights, exit time distributions, phase space topology prove to be important in the construction of kinetics. The concept of fractional kinetics enters a different area of applications, such as particle dynamics in different potentials, particle advection in fluids, plasma physics and fusion devices, quantum optics, and many others. New characteristics of the kinetics are involved to fractional kinetics and the most important are anomalous transport, superdiffusion, weak mixing, and others. The fractional kinetics does not look as the usual one since some moments of the distribution function are infinite and fluctuations from the equilibrium state do not have any finite time of relaxation. Different important physical phenomena: cooling of particles and signals, particle and wave traps, Maxwell's Demon, etc. represent some domains where fractional kinetics proves to be valuable.

PACS: 05.45.+b; 47.52.+j

Keywords: Chaos; Anomalous transport; Fractional kinetics

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Abbreviations: CTRW, continuous time random walk; FFPK, fractional Fokker–Planck–Kolmogorov equation; FK, fractional kinetics; FKE, fractional kinetic equation; GLG, generalization Lorentz gas; HIT, hierarchical island trap; MWE, Montroll–Weiss equation; RG, renormalization group; RGE, renormalization group equation; RGK, renormalization group of kinetics; SW, stochastic web; WRW, Weirstrass random walk.

0370-1573/02/$ - see front matter
PII: S0370-1573(02)00331-9

Contents

1. Introduction

Boltzmann was the first who applied dynamical methods for a kinetic description of physical objects, beginning with his "Kinetic Theory of Gases" in 1895. The main feature of Boltzmann's method is the application of a probabilistic manner of description of particles simultaneously with the original dynamical equations. This type of approach has become a standard one for large system evolution and it is a crucial attribute to any kind of kinetic theory. Despite the fact that the Boltzmann equation was non-linear with respect to the distribution function, later linear types of kinetic equations were derived, known now as balance or master equations. The main idea of the derivation of a kinetic equation is to reduce the number of variables of complex systems by applying some assumption of the statistical or probabilistic type. Diffusional equations can be considered as a simple example of kinetic equations, and their appearance is linked to the names of M. Smoluchowski, A. Einstein, M. Planck, and A. Kolmogorov. The presence of a probabilistic element in the derivation of kinetic equations results in a possibility of another way of getting kinetics by a rigorous introduction of some stochastic process relevant to the described phenomena. As an elementary example, one may consider the Gaussian process for a particle, wandering in a random media, with a belief that this is the adequate construction of particle motion. As a result, the diffusion equation for a particle's distribution appears in a natural way as a consequence of the corresponding stochastic process.

We use this trivial example to comment that the kinetic equation for real dynamical systems appears as a compromise between two alternative types of descriptions: dynamical and statistical. Kinetic equations do not describe full dynamics, and some features of the dynamics can never be obtained from the kinetics. At the same time, kinetics always consists of constraints that can contradict the dynamics and can exclude applicability of the corresponding stochastic process for some domains of variables and parameters. We will provide many such examples in the corresponding sections.

Dynamical chaos brings a new and immense area of research to the arena of the kinetic description of physical objects. Applying a notion of "randomness" to the trajectories of chaotic dynamical systems, we have to admit that a kinetic description of "typical" chaos is at the beginning of its development. Chaotic dynamics possesses a number of peculiar features that require a new approach in kinetics, in addition to the known tools, adjusted to complicated fractal and multi-fractal structures of phase space. Fractional kinetics (FK) (and, as a result, anomalous transport) for Hamiltonian systems with dynamical chaos is the subject of this review. Although some kind of fractional kinetics can be associated with the Lévy process, the real kinetics for even fairly simple models of chaos can be too far from the Lévy process, preserving only some of its features.

Considering trajectories of dynamical systems in phase space, we encounter very complicated and not completely known topology even for the simplest cases with 1 1/2 or 2 degrees of freedom. The phase space is non-uniform and consists of domains of chaotic dynamics (stochastic sea, stochastic webs, etc.) and islands of regular quasi-periodic dynamics. Other topological objects such as cantori and stable/unstable manifolds are also important to the study of kinetics. Trajectories, being considered as a kind of stochastic process, do not behave like well known Gaussian, Poissonian, Wiener, or other processes. The dynamics is strongly intermittent and a *topological zoo* imposes a type of kinetics with a wide spectrum of possibilities. Their classification has been started only recently, and many specific and fine properties of the dynamics should be understood in the way of construction of an adequate kinetics of chaos (for the earlier and some recent related reviews see, for example: Montroll and Shlesinger, 1984; Shlesinger, 1988; Bouchaud and Georges, 1990; Isichenko, 1992; Meiss, 1992; Wiggins, 1992; Shlesinger et al., 1993; Majda and Kramer, 1999; Bohr et al., 1998).

The goal of this review is to present existing models of fractional kinetics and their connection to dynamical models, phase space topology, and other characteristics of chaos. Section 2 consists of preliminaries of the dynamics of chaotic motion. It describes some typical structures of phase space and introduces important characteristics of chaotic dynamics: distribution of Poincaré recurrences and sticky domains. Section 3 consists of preliminaries for kinetics and basic constraints for the FPK equation, that we call *Kolmogorov condition*. Section 4 describes the important Lévy-type random process, its fundamental properties and generalizations. In Sections 5–8 we consider different aspects of FKE starting from its derivation, conditions of applicability, generalizations, probabilistic analogs, and concluding by the *renormalization group of kinetics*. Sections 9 and 10 are related to specific dynamical properties, *dynamical traps*, that influence the transport and make it anomalous and imposes the fractional type of kinetics. All other Sections 11–16 are different applications of FK: Statistical laws in the cases of the presence of dynamical traps (Section 11); advection described by FK (Section 12) and related issues in plasma physics (Section 13); particle dynamics in potentials with different symmetry (Section 14). Special application of FK is in Section 15 where we discuss systems with zero Lyapunov exponent but, nevertheless, with a kind of randomness called *pseudochaos*. Particularly, such a situation occurs in different polygonal billiards, Lorentz gas, and the vector field lines behavior. Section 16 can be considered as an introduction to FK properties of quantum systems. The appendices have useful formulas of fractional calculus for the reader's convenience. There is no necessity to be familiar with fractional calculus for reading the review.

2. Preliminaries: dynamics

2.1. General comments

In this section we consider some features of the chaotic dynamics which are relevant to the kinetic description of a system. Equations of the Hamiltonian dynamics are

$$\dot{\mathbf{p}} = -\partial H/\partial \mathbf{q}, \quad \dot{\mathbf{q}} = \partial H/\partial \mathbf{p} \ , \tag{1}$$

with generalized momentum $\mathbf{p} = (p_1, \ldots, p_N)$, coordinate $\mathbf{q} = (q_1, \ldots, q_N)$, and Hamiltonian $H = H(\mathbf{p}, \mathbf{q}, t)$ which may depend on time t. The phase space of the system $(\mathbf{p}, \mathbf{q})$ has $2N$ dimensions, but even for $N = 1$ and time-dependent Hamiltonian, the systems of a general type have

very complicated structure of phase space, where the domains of regular (non-chaotic) dynamics alternate with the domains of erratic (chaotic) motion. It may not be so unexpected that kinetics in the domains of chaotic motion is not independent on what is happening near the chaos-order borders.

The task of describing the dynamics that, in some sense, has a seed of randomness, is not simple and is not clear in many aspects. For example, all topological elements of phase space and their properties are not known and there is no complete scheme of the classification of the trajectories. We know about the existence of invariant curves (tori), stable/unstable manifolds, singular points or surfaces, and cantori, but even a zero measure set of points of phase space can drastically change the kinetics and transport properties of a particle. The concept of the neglecting of zero measure sets is typical for the ergodic theory of dynamical systems. The importance of some zero measure sets for the kinetics goes far beyond the so-called "random phase" hypothesis widely used to describe the transition from dynamics to kinetics. It also means a necessity of the search for new classes of universality of kinetics, depending on the type of chaotic dynamics and phase space topology, and a necessity of the careful selection of a specific type of random process to describe the chaotic dynamics. We will demonstrate for such popular examples as the standard map or Sinai billiard how some zero measure sets and fine features of the phase space topology come into account for the kinetic equations.

2.2. Mapping the dynamics

Despite the fact that trajectories can be arbitrarily smooth, they can also display different fractal features "extracted" after applying to the trajectories a specific algorithm. A trajectory can be considered not as a solution of the original Hamiltonian equations but, instead, as a solution of a *map* which is a set of discrete equations. This map should represent the trajectories, but part of the information about dynamics is hidden in the structure of the map. Maps are convenient for different simplifications of the dynamical equations. Below we show few examples of mapping the dynamics.

2.2.1. Poincaré map

The Poincaré map is a set of points in phase space $(\mathbf{p},\mathbf{q})$ obtained from the intersection of directed trajectories with a hypersurface. The intersection points define a set of time instants $\{t_j\}$ of intersection. The map can be presented in the form

$$(\mathbf{p}_{n+1},\mathbf{q}_{n+1}) = \hat{T}_n(\mathbf{p}_n,\mathbf{q}_n), \quad t_{n+1} = \hat{f}(t_n) \tag{2}$$

with an appropriate time-shift operator $\hat{T}_n$ and function $\hat{f}$. For 1 1/2 or 2 degrees of freedom and finite motion, the closure of the set $(\mathbf{p}_i,\mathbf{q}_i)$ is an invariant curve or invariant torus if the dynamics is quasi-periodic, and it has randomly distributed points (stochastic sea) if the dynamics is chaotic. For a generic situation, the Poincaré map forms a stochastic sea with infinite numbers of implanted islands. The dynamics in the islands is isolated from the dynamics in the stochastic sea and vice versa. The dynamics inside the islands corresponds to the quasi-periodic trajectories as well as to the isolated narrow stochastic *layers* with stochastic trajectories inside the layers. This intricate, sandwich-type structure of phase space is at the heart of fractional dynamics. More about the Poincaré map is in Arnold (1978), and Lichtenberg and Lieberman (1983).

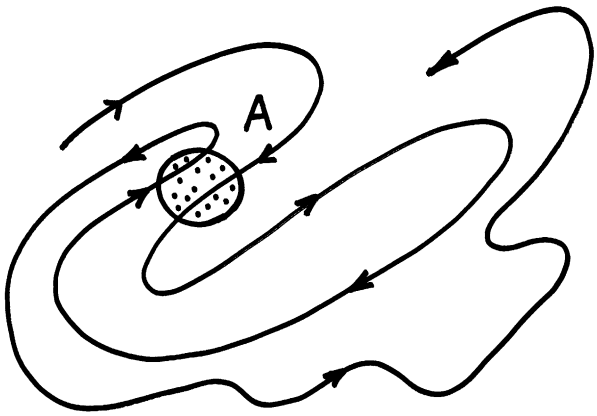

Fig. 1. Poincaré recurrences to the domain A.

The Poincaré map is the most effective for 1 1/2 or 2 degrees of freedom. The second equation in (2) for the mapping time is a source of serious complications unless

$$\Delta t_n = t_{n+1} - t_n = \text{const.} \tag{3}$$

for all n, or is approximately constant. The Poincaré map will be considered in more detail later.

2.2.2. Poincaré recurrences

Poincaré recurrences, or simply *recurrences*, is one of the most important notions for the kinetics. For a long time since the famous discussion between Boltzmann and Zermelo, the Poincaré recurrences did not play a serious constructive role in kinetics, although many derivations of the transport equations involve the properties of periodic orbits and their distributions (see for example in Bowen, 1972; Ott, 1993; Dorfman, 1999). Consider a domain A in phase space and a typical set of points $\{\mathbf{p}, \mathbf{q} \in A\}$. The Poincaré theorem of recurrences states that for any initial condition of the typical set, trajectories will repeatedly return to A if the dynamics is area preserving and confined. Let $\{t_j^-\}$ be a set of time instants when a trajectory leaves A, and let $\{t_j^+\}$ be the set of time instants when a trajectory enters A (Fig. 1). The set

$$\{\tau_j^{(\text{rec})}\}_A = \{t_j^- - t_{j-1}^-\}_A \tag{4}$$

is the set of recurrences. We call $\{\tau_j^{(\text{rec})}\}$ the recurrence time at j-step, or the jth cycle. There are some other characteristics related to the recurrences.

Exit (escape) times can be introduced as a time interval between the adjacent entrances to A and exits from A:

$$\{\tau_j^{(\text{esc})}\}_A = \{t_j^- - t_j^+\}_A \ . \tag{5}$$

Similarly, one can introduce time intervals $\{\tau_j^{(\text{ext})}\}$ that a trajectory spends outside of A during one cycle:

$$\{\tau_j^{(\text{ext})}\}_A = \{t_j^+ - t_{j-1}^-\}_A \ . \tag{6}$$

There is an evident connection:

$$\tau_j^{(\mathrm{rec})} = \tau_j^{(\mathrm{esc})} + \tau_j^{(\mathrm{ext})} \ . \tag{7}$$

The set of recurrence times $\{\tau_j^{(\mathrm{rec})}\}$ forms a specific mapping of trajectories. The sequence $\{\tau_j^{(\mathrm{rec})}\}$ can have fractal or multifractal features (Zaslavsky, 1995; Afraimovich, 1997; Afraimovich and Zaslavsky, 1997, 1998). It also can be characterized by the probability distribution function $P_{\mathrm{rec}}(\tau; A)$. These properties of recurrences will be considered in Section 2.3. Similarly, one can consider the mapping for $\{\tau_j^{(\mathrm{esc})}\}$ and $\{\tau_j^{(\mathrm{ext})}\}$.

2.2.3. Standard map

The standard map is also known as the Chirikov–Taylor map (Chirikov, 1979). It is written for momentum p and coordinate x as

$$p_{n+1} = p_n + K \sin x_n, \quad x_{n+1} = x_n + p_{n+1} \ , \tag{8}$$

where K is a parameter that characterizes the force amplitude, and (p, x) are defined either on a torus $(-\pi < p < \pi;\ -\pi < x < \pi)$, or on a cylinder $(-\infty < p < \infty;\ -\pi < x < \pi)$, or in infinite space $(-\infty < p < \infty;\ -\infty < x < \infty)$. The standard map has numerous applications in accelerator physics, plasma physics, condensed matter, etc. (see for example Chirikov, 1979; Lichtenberg and Lieberman, 1983; Sagdeev et al., 1988). The map describes a periodically kicked rotor as well as many other systems that can be reduced to the similar equation. Eqs. (8) are of the difference type, while the original differential equations are generated by the Hamiltonian:

$$H = p^2/2 - K \cos x \sum_{m=-\infty}^{\infty} \delta(t - m) \ . \tag{9}$$

Map (8) has chaotic trajectories for any K. Domains of chaotic trajectories are bounded in the p-direction for $K < K_c$, forming a set of stochastic layers, and are unbounded in the p-direction for $K \geqslant K_c$ forming the so-called *stochastic sea*. The value of K_c is known: $K_c = 0.9716\ldots$ (see Greene, 1979; Greene et al., 1981; Kadanoff, 1981; MacKay, 1983). Fig. 2 shows how complex is the phase space structure of a "typical" dynamical system.

2.2.4. Web map

The web map appearing in Zaslavsky et al. (1986) for the study of charged particle dynamics in a constant magnetic field and perpendicular electric wave packet. The Hamiltonian corresponds to a periodically kicked linear oscillator:

$$H = \frac{1}{2}(\dot{x}^2 + \omega^2 x^2) - K_0 \cos x \sum_{m=-\infty} \delta(t - mT) \ . \tag{10}$$

For the resonant condition $\omega T = 2\pi/q$ and integer q, system (10) generates the so-called *stochastic web map* of the q-fold symmetry in phase space (see more in Zaslavsky et al., 1991). For $q = 4$, the corresponding map is

$$u_{n+1} = v_n, \quad v_{n+1} = -u_n - K \sin v_n \ , \tag{11}$$

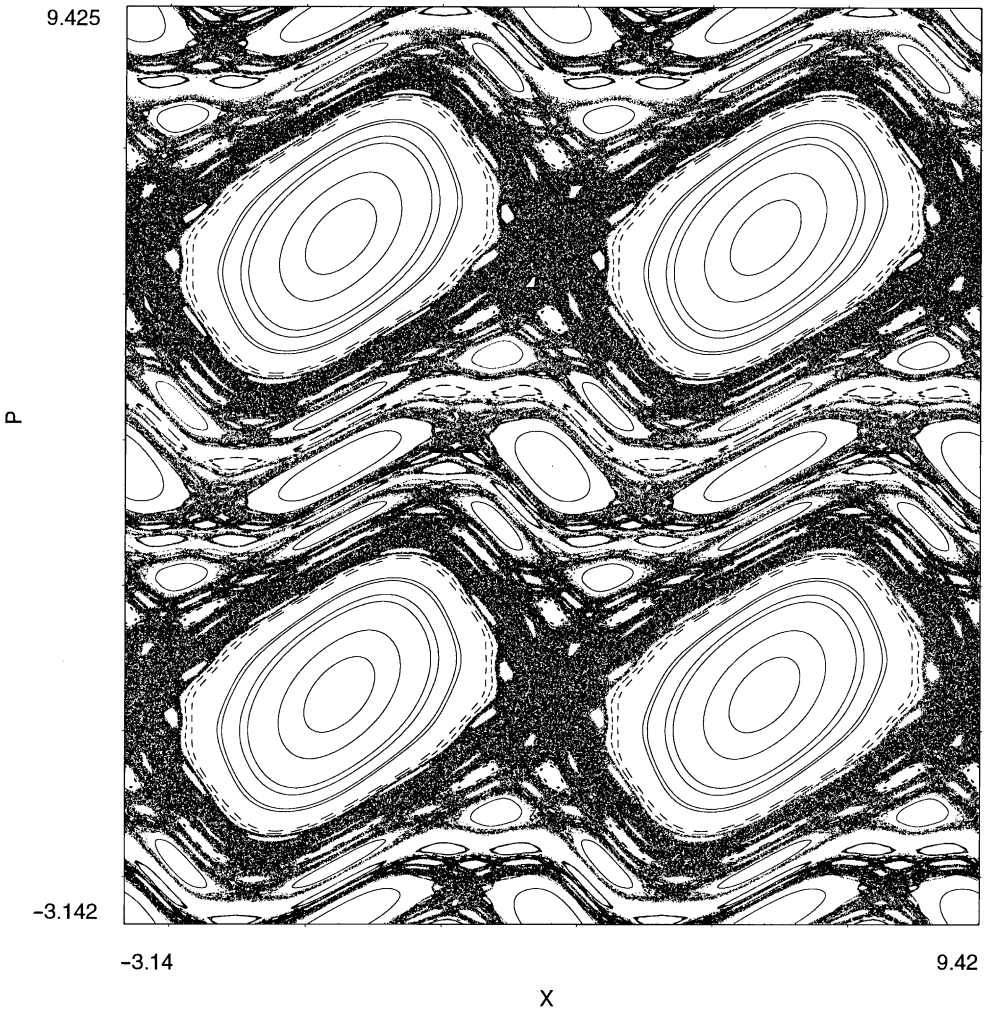

Fig. 2. Phase portrait of the standard map ($K = 1.0$).

where we put $\omega_0 = 1$, $K = 2K_0/\pi$, $u = \dot{x}$, $v = -x$. Again, as for the standard map, the region of consideration can be: a torus $(u, v) \in (-\pi, \pi)$; a cylinder $u \in (-\infty, \infty)$, $v \in (-\pi, \pi)$ or $u \in (-\pi, \pi)$, $v \in (-\infty, \infty)$; or infinite (cover) space $(u, v) \in (-\infty, \infty)$. The stochastic web tiles infinite space with a corresponding type of symmetry for any value of K (see Fig. 3). The channels of the web are exponentially narrow for $K \ll 1$ and narrow for $K \sim 1$ (Fig. 3, part (a)). For $K \gg 1$, the stochastic sea covers full phase space leaving holes for islands that are filled by the nested invariant curves and isolated stochastic layers (Fig. 3, part (b)).

The main property of the web map (11) is that its phase space topology includes an unbounded domain of chaotic dynamics for arbitrary value of the control parameter K, contrary to the standard map. This makes the web map important to study of the unbounded transport of particles and the transport dependence on the phase space topology.

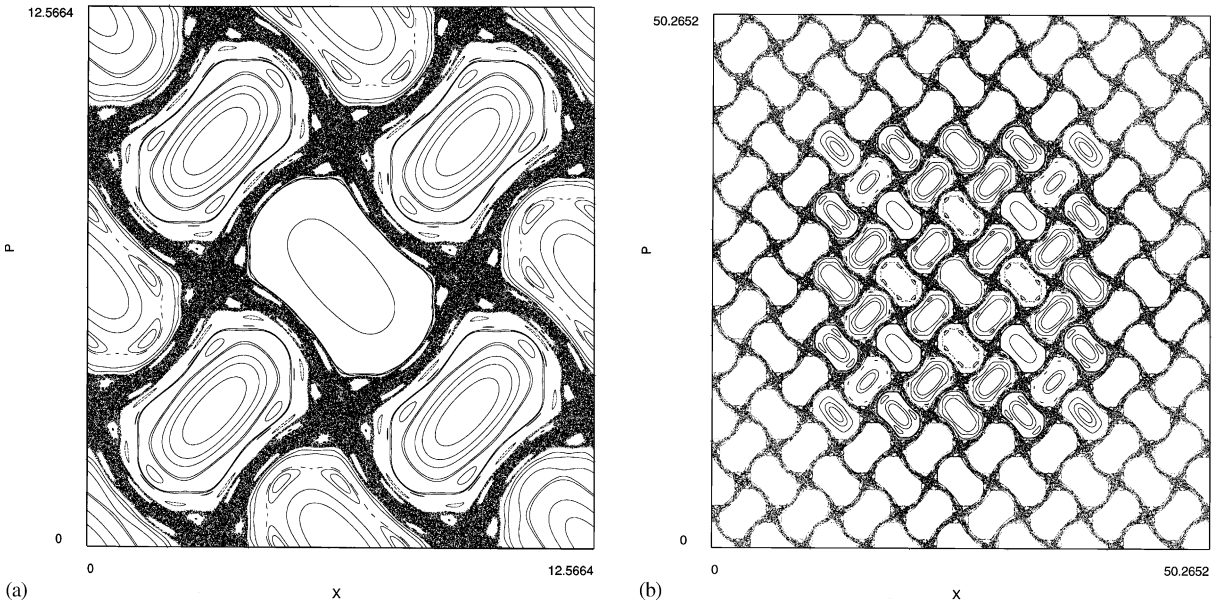

Fig. 3. Web map ($K = 1.5$): (a) phase portrait; (b) formation of the web with 4-fold symmetry in the cover space (periodically continued in the both directions).

2.2.5. Zeno map

The Greek philosopher Zeno of Elea (circa 450 B.C.), is credited with creating several famous paradoxes, one of which, the "Achilles paradox", was described by Aristotle in the treatise *Physics*. The paradox concerns a race between the fleet-footed Achilles (the Greek hero of Homer's *The Illiad*), and a slow-moving tortoise. The Achilles paradox postulates that as both start moving at the same moment, Achilles can never catch the tortoise, if the tortoise is given a head start. The Zeno map conveys this idea that Achilles can never catch the tortoise. It is a set of positions $\{x_j\}$, with Achilles initially at point x_0, and the tortoise at point x_1. When Achilles arrives at x_1, the tortoise achieves x_2 and evidently $x_2 - x_1 \ll x_1 - x_0$. When Achilles arrives at x_2, the tortoise is at x_3 and $x_3 - x_2 \ll x_2 - x_1$, and so on. There is always a finite distance between Achilles and the tortoise which is supposed to move with constant velocities. A solution for this paradox is in a special selection of the set of time instants t_j that generate the map $\{x_0, x_1, \ldots\}$. This solution shows that not any map of form (2) can represent dynamics which will adequately correspond to real trajectories or, simply, will adequately present the trajectories in phase space. This solution cautions us in the choice of a map that should replace real trajectories.

2.2.6. Perturbed pendulum

It seems that the perturbed pendulum problem first appeared in Landau and Lifshits (1976) in few modifications:

$$\ddot{x} + \omega_0^2 \sin x = \epsilon \omega_0^2 \sin(x - \nu t) \tag{12}$$

and

$$\ddot{x} + \omega_0^2 [1 + \epsilon \cos \nu t] \sin x = 0 \tag{13}$$

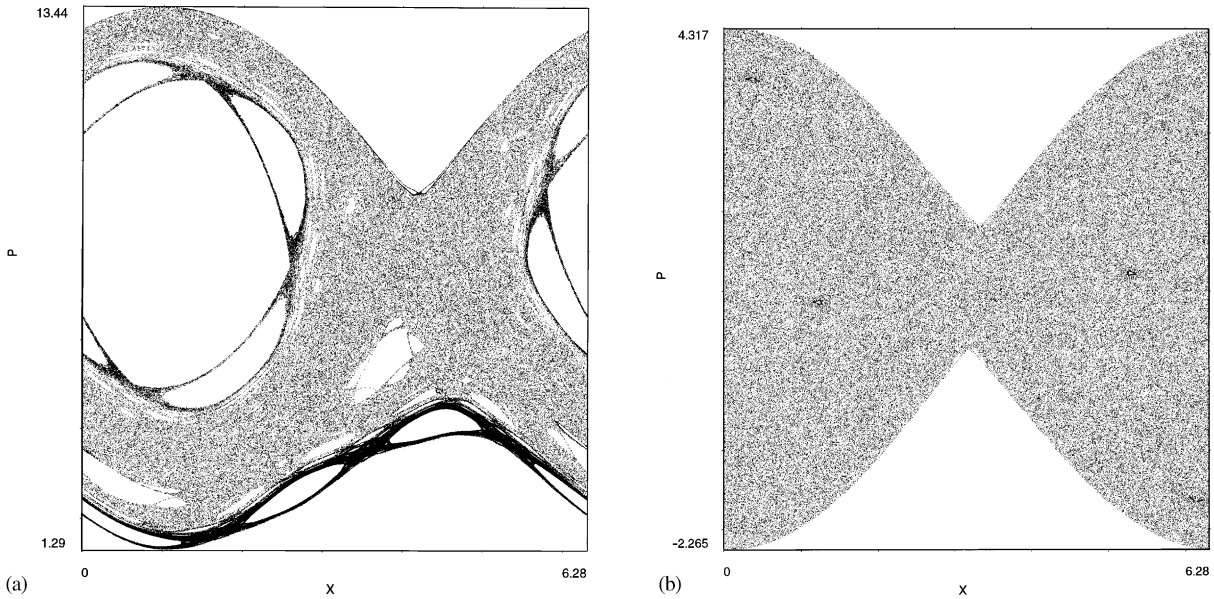

Fig. 4. Poincaré map for the perturbed pendulum: (a) example of a stochastic layer with a sticky trajectory ($\epsilon = 7.86055$; $\nu = 8.57$); (b) another example of the stochastic layer with different topology and almost invisible islands.

depending on the dynamics of the suspension point of the pendulum. The problem became a paradigm due to its numerous applications (see a review in Zaslavsky et al., 1991). The Poincaré map appears as a set of points taken with time interval $2\pi/\nu$, i.e. at $t_j = 0,\ 1 \cdot 2\pi/\nu,\ 2 \cdot 2\pi/\nu, \ldots$.

In both Eqs. (12) and (13) there are two dimensionless parameters that control the phase space topology: ϵ and ν/ω_0. Two examples in Fig. 4 show very different structures of the so-called *stochastic layers*, elongated along x zones of chaotic dynamics. Transport of particles along x is different and depends on the topology, i.e. the transport is sensitive to the parameters ϵ and ν/ω_0. Particularly, some dark regions indicate zones where a trajectory can spend a longer time than in other zones. This phenomenon is known as *stickiness* and it will be discussed later.

2.3. Distribution of recurrences

Boltzmann was the first who estimated the mean recurrence time for a gas model of particles and showed that it is super-astronomically large. Boltzmann is also credited with the important comment that the Poincaré theorem of recurrences does not involve anything related to the time duration of the cycles. The important result on this issue was obtained by Kac (1958). Let $P(\tau; A)$ be the probability density to return to A at time $\tau \in (\tau, \tau + \mathrm{d}\tau)$. For small $\Gamma(A)$, $P(\tau; A)$ should be proportional to $\Gamma(A)$ if the system dynamics is area-preserving and bounded. More accurately, one can introduce a probability density

$$P_{\mathrm{rec}}(\tau) = \lim_{\Gamma(A) \to 0} \frac{1}{\Gamma(A)} P(\tau; A) \tag{14}$$

which exists and is normalized

$$\int_0^\infty P_{\rm rec}(\tau)\,{\rm d}\tau = 1 \ . \tag{15}$$

The mean recurrence time is

$$\tau_{\rm rec} = \langle\tau\rangle = \int_0^\infty \tau P_{\rm rec}(\tau)\,{\rm d}\tau \ . \tag{16}$$

Due to the Kac lemma

$$\tau_{\rm rec} < \infty \tag{17}$$

for area-preserving and bounded dynamics (see also in Meiss, 1997; Cornfeld et al., 1982).

Kac's works on recurrences were the only ones with a new significant result since a very long time after Poincaré, Zermelo, and Boltzmann. The lack of interest in this subject could be explained by the huge value of $\tau_{\rm rec}$ for the considered statistical models, so that there was no realistic possibility to measure either $\tau_{\rm rec}$ or any effect related to the finiteness of $\tau_{\rm rec}$. The situation has been changed after the discovery of chaos and its practical implementations.

Chaos occurs in systems with a few degrees of freedom, starting from 1 1/2 degrees, and the time of Poincaré cycles becomes comparable to a characteristic time of the system. It was shown by Margulis (1969, 1970) that for systems with a uniform mixing property (Anosov-type systems):

$$P(\tau) = \frac{1}{\tau_{\rm rec}} \exp\left(-\frac{\tau}{\tau_{\rm rec}}\right) \tag{18}$$

with

$$\tau_{\rm rec} = 1/h \ , \tag{19}$$

where h is Kolmogorov–Sinai entropy, or a characteristic rate of the divergence of close trajectory:

$$\delta r(t) = \delta r(0)\exp(ht) \tag{20}$$

with $\delta r(0) \to 0$ as an initial distance between any two trajectories, and $\delta r(t)$ as their distance at time t (see also in Zaslavsky, 1985).

The exponential distribution of recurrences (18) is not the only possible one, and some simulations showed the possibility of an algebraic asymptotic distribution (Chirikov and Shepelyansky, 1984; Karney, 1983):

$$P_{\rm rec}(\tau) \sim 1/\tau^\gamma \quad (\tau \to \infty) \ . \tag{21}$$

We call γ the *recurrence exponent*. It follows from the Kac lemma (17) that

$$\gamma > 2 \tag{22}$$

(see also in Meiss, 1997; Zaslavsky and Tippett, 1991).

Algebraic decay of $P_{\rm rec}(\tau)$ is a result of a non-uniformity of phase space, and can be conjectured as a universal phenomenon for typical Hamiltonian systems with a stochastic sea and islands (Zaslavsky et al., 1997; Zaslavsky, 1999, 2002). A more accurate statement of this conjecture is that there exists at least a dense set of values of K for which

$$1/\tau^{\gamma_2} < P_{\rm rec}(\tau) < 1/\tau^{\gamma_1} \quad (\tau \to \infty) \tag{23}$$

with finite γ_1, γ_2, and a non-uniform dependence of

$$\gamma_{1,2} = \gamma_{1,2}(K) \ , \tag{24}$$

where K is any control parameter of the system. This property will be discussed more in the corresponding section and, particularly, it will be shown how the algebraic behavior of $P_{\mathrm{rec}}(\tau)$ is related to fractal and multifractal properties of dynamics.

2.4. Singular zones

The studies of the chaotic dynamics by mathematicians and physicists was different from the very beginning. While mathematicians were attracted by problems that permit a rigorous consideration (Anosov systems, Smale horseshoe Sinai billiard), physicists considered mainly the problems of an important physical meaning such as the stability of particles in accelerators or the destruction of magnetic surfaces in toroidal plasma confinement. We comment on this in order to continue the treatment of some physically based conjectures which are directly related and important for physical problems but still do not have a rigorous foundation. As a part of this discussion is the origin of fractional kinetics which prove to appear due to the non-uniformity of phase space.

There are different reasons for the phase space non-uniformity. The main reason can be related to the existence of islands in any kind of domain of chaotic dynamics with a smooth Hamiltonian. The islands can be of different origin (resonances, bifurcations, etc.), and the phase space properties near the islands' boundaries are not well known (Meiss, 1986, 1992; Meiss and Ott, 1985, 1986; Rom-Kedar, 1994; Melnikov, 1996; Zaslavsky et al., 1997; Rom-Kedar and Zaslavsky, 1999). In addition to the islands, let us mention the so-called cantori-zero-dimension periodic sets that surround islands, having an infinite number of the cantor-set type holes and creating penetrable barriers for transport (Percival, 1980; Meiss et al., 1983; Hanson et al., 1985).

Here we would like to present only two examples. Both are related to the standard map (8), but correspond to different values of K. Fig. 5 shows a "griddle"-type vicinity of an island. The island appears (and can disappear) as a result of a bifurcation from a parabolic point (Newhouse, 1977; Melnikov, 1996; Karney, 1983). The darkness of some domains in Fig. 5 indicates a long-term stay of the trajectory in the domain. Such phenomenon is called stickiness (Karney, 1983; Meiss and Ott, 1985; Hanson et al., 1985). Another pattern of the stickiness is shown in Fig. 6 where a hierarchy of islands appears for a special value of K (Zaslavsky et al., 1997).

A general feature of Hamiltonian systems is that their phase space is full of sticky domains which we call *singular zones*. They can be small and invisible in the Poincaré maps, they depend on the control parameters, and they can be modified in many ways, but their consideration is unavoidable for a general theory and they influence kinetics and transport. For some earlier works on non-Hamiltonian systems see Geisel (1984) and Geisel and Nierwetberg (1984).

2.5. Sticky domains and escapes

The kind of stickiness imposes the type of anomalous transport. It can be that the sticky domain does not move with time or it does move in phase space. A trajectory that enters the sticky area escapes from it after a very long time and this part of the trajectory represents a "ballistic" one that can be characterized by an almost regular dynamics with almost constant acceleration (stickiness to

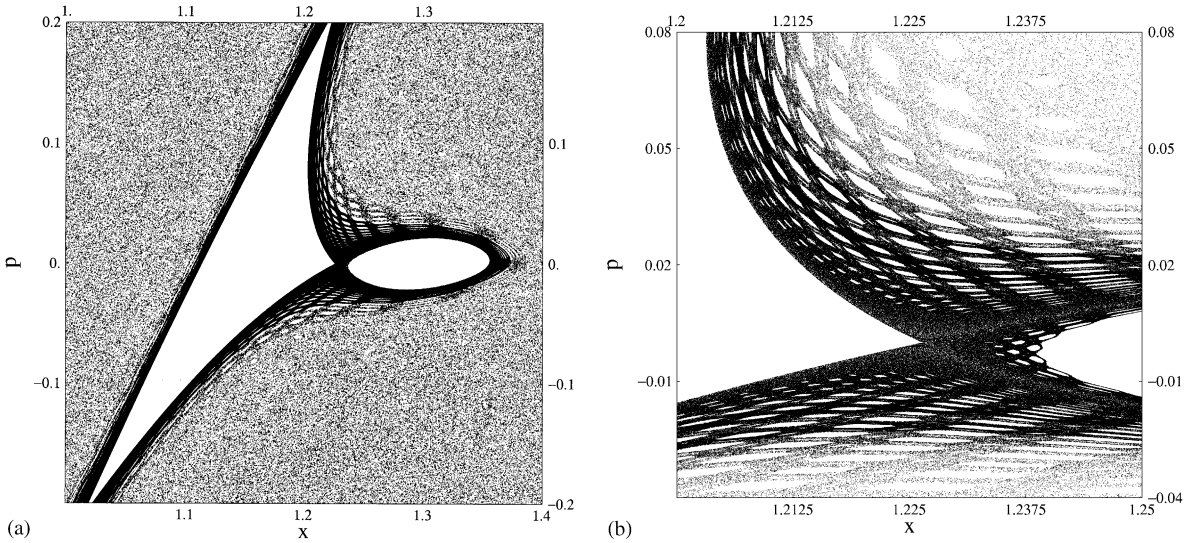

Fig. 5. A "net trap" example of a singular zone for the standard map with $K = 6.9009$ after 10^9 iterations: (a) an island that is close to the bifurcation value of K; (b) magnification of the vicinity of the island.

the accelerator mode island), or almost constant velocity (stickiness to the ballistic type island; see Section 9). These are also sticky domains of other than accelerator or ballistic type (see Section 9).

The origin of the stickiness does not have a universal scenario. A passage through the cantori was described in Meiss et al. (1983), Hanson et al. (1985), and Meiss and Ott (1985). Almost isolated stochastic layers were considered in Petrovichev et al. (1990), Rakhlin (2000), and Zaslavsky (2002); islands-around-islands scenarios were considered in Meiss (1986, 1992), Zaslavsky et al. (1997), and Rom-Kedar and Zaslavsky (1999); even one single marginal fixed point can generate a stickiness and anomalous transport (Artuso and Prampolini, 1998). One can introduce a probabilistic description of the stickiness using escape rate probability $P_{\rm esc}(t)\,{\rm d}t$ that shows probability for a particle to escape a domain within the interval $(t, t + {\rm d}t)$, normalized as

$$\int_0^\infty {\rm d}t\, P_{\rm esc}(t) = 1 \ . \tag{25}$$

If asymptotically

$$P_{\rm esc}(t) \sim {\rm const.}/t^{\gamma_{\rm esc}} \quad (t \to \infty) \tag{26}$$

and there is the only sticky domain with the *escape exponent* $\gamma_{\rm esc}$, then one may expect the following relation:

$$\gamma = \gamma_{\rm esc} \ , \tag{27}$$

where $\gamma = \gamma_{\rm rec}$ (see (21)) is the exponent of distribution of recurrences, and consequently

$$P_{\rm rec}(t) \sim P_{\rm esc}(t) \quad (t \to \infty) \ . \tag{28}$$

If there is more than one singular zone with different values of $\gamma_{\rm esc}$, then instead of (27), we have

$$\gamma_{\rm rec} = \min \gamma_{\rm esc} \quad (t \to \infty) \tag{29}$$

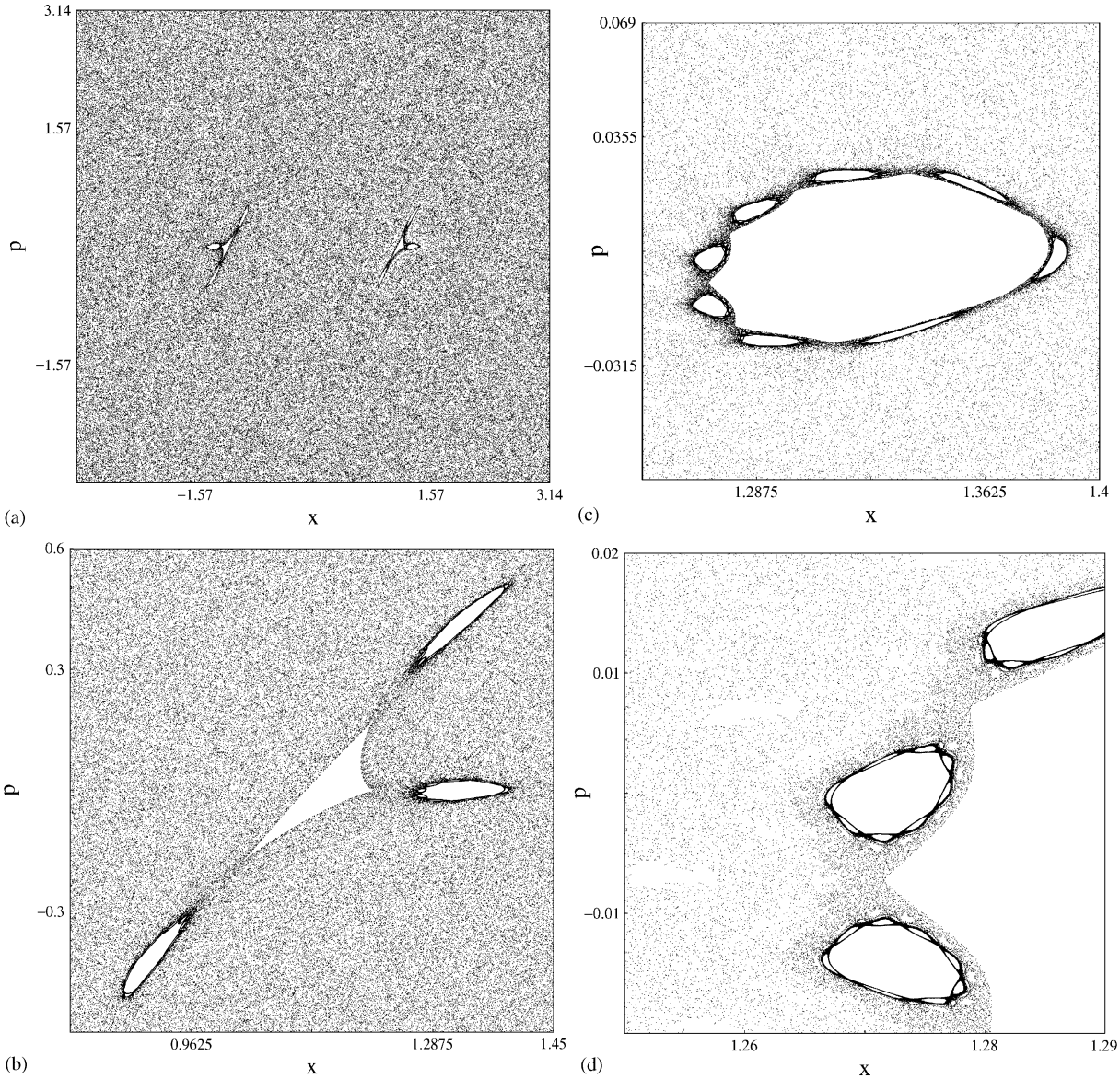

Fig. 6. Hierarchy of islands 2–3–8–8–8–... for the standard map for $K = 6.908745$ (Zaslavsky et al., 1997).

since the minimum value of γ_{esc} defines the largest time asymptotics. A numerical example for the standard map of the equivalence of two asymptotics (28), Fig. 7. After averaging over the number of trajectories, the difference between P_{rec} for the probability density to return to the square A, and P_{targ} of the probability density to leave A and reach first time B, is indistinguishable. Both domains A and B are in the stochastic sea and the majority of trajectories arrives first time to B or returns first time to A after a fairly short time corresponding to law (18). After $t \gg \tau_{rec}$ there is a fairly small number of such trajectories. This group consists of trajectories that are visiting a singular zone

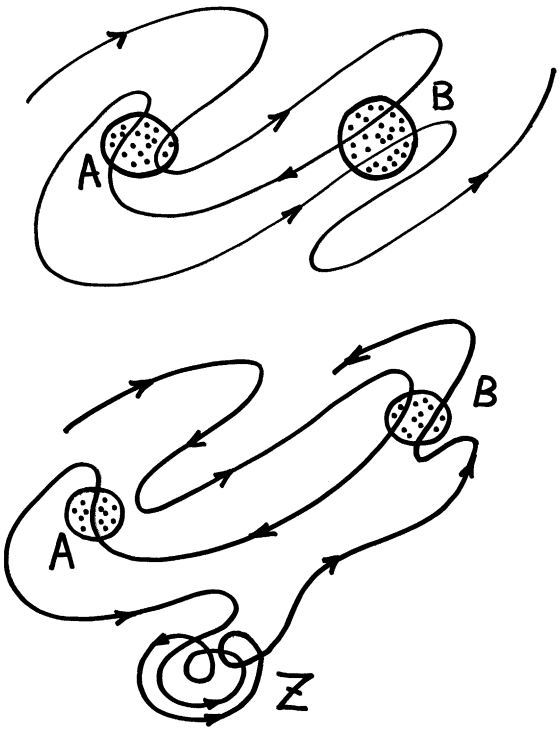

Fig. 7. Targeting for the standard map: (a) location of the starting domain A_{st} and the targeting domain A_{tg} (both as black squares); (b) distribution of the $P_{rec}(t)$. Slope 1 is -2.19, slope 2 is -4.09. Distribution $P_{targ}(t)$ is almost indistinguishable from $P_{rec}(t)$. The data obtained after averaging over 3.22×10^9 initial conditions (Zaslavsky and Edelman, 2000).

(see Fig. 8) on their way back to A (or to the target B). Just these delayed trajectories are responsible for the asymptotic behavior of $P_{rec}(t)$ and $P_{esc}(t)$.

This simple comment implies a connection between singular zones and transport, that will be discussed in different following sections.

3. Preliminaries: kinetics

3.1. General comments

A splitting of the dynamics into two different time-scales: short Δt and long τ_{col}, is in the heart of the kinetic description of systems. Short time-scale is typically associated with the duration of a "collision", and long time-scale τ_{col} corresponds to the interval between adjacent collisions. The "collision" should be considered in a broad sense: it may be a kind of essential change of energy or momentum due to an interaction with the external field, or a change of variables due to the real collision, or a strong change of the adiabatic invariants. Another important feature of the derivation of the kinetic equations is a reduction of the number of variables by an averaging over *fast variable*(*s*):

$$P(p,t) \equiv \langle\langle P(p,q;t) \rangle\rangle \ , \tag{30}$$

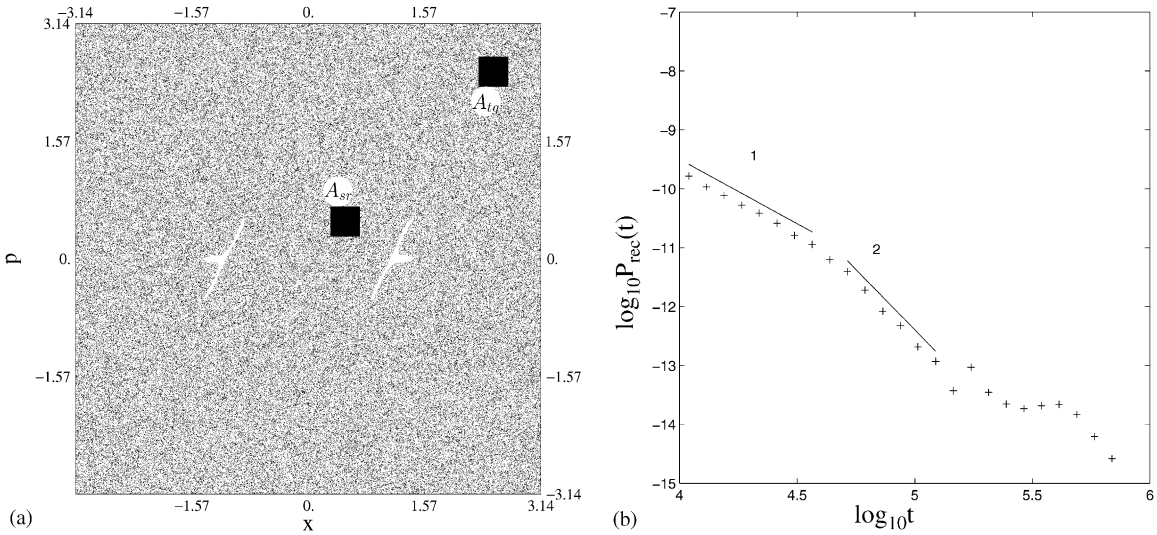

Fig. 8. Targeting without (top) and with (bottom) a singular zone Z. A trajectory escapes from the domain A and arrives first time to the domain B.

where P is probability density, (p,q) are generalized momentum and coordinate correspondently, q is considered as the fast variable and double angular brackets denote averaging over q (see more for the Hamiltonian chaotic dynamics in Zaslavsky, 1985). A typical time-scale connection reads:

$$t \gg \tau_{\mathrm{col}} \gg \Delta t \ . \tag{31}$$

Although conditions (31) are the most typical ones to obtain the kinetic equation for $P(p,t)$, the existence of singular zones and sticky domains imposes a number of deviations from the usual scheme. In this section will derive a regular diffusion equation and compare the conditions of its validity to the conditions of the derivation of the so-called *fractional kinetic equation* (FKE). We will also make a comparison between the kinetics derived from the dynamics with the kinetics obtained in the probabilistic way, the so-called Continuous Time Random Walk (CTRW) (see Montroll and Weiss, 1965; Montroll and Shlesinger, 1984; Shlesinger et al., 1987; Weiss, 1994). We will also indicate the conditions when the CTRW fails due to physical reasons.

3.2. Fokker–Planck–Kolmogorov equation (FPK)

The FPK equation was first obtained by Fokker (1914) and Planck (1917). Kolmogorov (1938) derived a kinetic equation using a special scheme and conditions that are important for understanding some basic principles of kinetics. Let $W(x,t;x',t')$ be a probability density of having a particle at the position x at time t if the particle was at the position x' at time $t' \leqslant t$. A chain equation of the Markov-type process can be written for $W(x,t;x',t')$:

$$W(x_3,t_3;x_1,t_1) = \int \mathrm{d}x_2\, W(x_3,t_3;x_2,t_2) W(x_2,t_2;x_1,t_1) \ . \tag{32}$$

A typical assumption for W is its time uniformity, i.e.:

$$W(x,t;x',t') = W(x,x';t-t') \ . \tag{33}$$

Consider the evolution of $W(x,x';t-t')$ during an infinitesimal time $\Delta t = t' - t$ and use the expansion

$$W(x,x_0;t+\Delta t) = W(x,x_0;t) + \frac{\partial W(x,x_0;t)}{\partial t}\Delta t + \cdots \ . \tag{34}$$

Eq. (34) is valid providing the limit:

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\{W(x,x_0;t+\Delta t) - W(x,x_0;t)\} = \frac{\partial W(x,x_0;t)}{\partial t} \tag{35}$$

has a sense. The existence of limit (35) for $\Delta t \to 0$ imposes specific physical constraints that will be discussed in this section later.

Let us now introduce a new notation:

$$P(x,t) \equiv W(x,x_0;t) \ , \tag{36}$$

where the initial coordinate x_0 is omitted. With the help of Eqs. (32)–(34), we can transform Eq. (35) into

$$\frac{\partial P(x,t)}{\partial t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int \mathrm{d}y\, W(x,y;\Delta t)P(y,t) - P(x,t)\right\} \ . \tag{37}$$

The first important feature in the derivation of the kinetic equation is the introduction of two distribution functions $P(x,t)$ and $W(x,y;\Delta t)$ instead of the one $W(x,p;x',p')$. The function $P(x,t)$ will be used for $t\to\infty$ or, more accurately, for t which satisfies (31), where τ_{col} is not defined yet. In this situation $W(x,x_0;t)$ does not depend on the initial condition x_0 and this explains notation (36). Contrary to $P(x,t)$, $W(x,y;\Delta t)$ defines the transition during very short time $\Delta t \to 0$. For $\Delta t = 0$ it should be no transition at all if the velocity is finite, i.e.:

$$\lim_{\Delta t\to 0} W(x,y;\Delta t) = \delta(x-y) \ . \tag{38}$$

Following this restriction we can use the expansion over the δ-function and its derivatives (Zaslavsky, 1994a, b), i.e.:

$$W(x,y;\Delta t) = \delta(x-y) + A(y;\Delta t)\delta'(x-y) + \frac{1}{2}B(y;\Delta t)\delta''(x-y) \ , \tag{39}$$

where $A(y;\Delta t)$ and $B(y;\Delta t)$ are some functions. The prime denotes the derivative with respect to the argument, and we consider the expansion up to the second order only.

Distribution $W(x,y;\Delta t)$ is called *transfer probability* and it satisfies two normalization conditions:

$$\int W(x,y;\Delta t)\,\mathrm{d}x = 1 \tag{40}$$

and

$$\int W(x,y;\Delta t)\,\mathrm{d}y = 1 \ . \tag{41}$$

The coefficients $A(x;\Delta t)$ and $B(x;\Delta t)$ have a fairly simple meaning. They can be expressed as moments of $W(x,y;\Delta t)$:

$$A(y;t)=\int \mathrm{d}x(y-x)W(x,y;\Delta t)\equiv \langle\langle \Delta y\rangle\rangle \ ,$$

$$B(y;t)=\int \mathrm{d}x(y-x)^2W(x,y;\Delta t)\equiv \langle\langle (\Delta y)^2\rangle\rangle \ . \tag{42}$$

In a similar way, coefficients for the higher orders of the expansion of $W(x,y;\Delta t)$ can be expressed through the higher moments of W.

Integration of (39) over x does not provide any additional information due to (40), but integrating over y and using (41) give

$$A(y;\Delta t)=\frac{1}{2}\frac{\partial B(y;\Delta t)}{\partial y} \tag{43}$$

or applying notations (42)

$$\langle\langle \Delta y\rangle\rangle=\frac{1}{2}\frac{\partial}{\partial y}\langle\langle (\Delta y)^2\rangle\rangle \ . \tag{44}$$

Expressions (43) and (44) were first obtained in Landau (1937) as a result of the *microscopic reversibility*, or detailed balance principle. In Landau (1937), dynamical Hamiltonian equations were used for (44), while here we use expansion (39) and "reversible" normalization (41). The final step is an assumption that we call *Kolmogorov conditions*: there exist limits:

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle\langle \Delta x\rangle\rangle\equiv \mathscr{A}(x) \ ,$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle\langle (\Delta x)^2\rangle\rangle\equiv \mathscr{B}(x) \ ,$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle\langle (\Delta x)^m\rangle\rangle=0 \quad (m>2) \ . \tag{45}$$

It is due to the Kolmogorov conditions that irreversibility appears at the final equation. Now it is just formal steps. Substituting (39), (42), and (44) into (37) gives

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}(\mathscr{A}P(x,t))+\frac{1}{2}\frac{\partial^2}{\partial x^2}(\mathscr{B}P(x,t)) \tag{46}$$

which is the equation derived by Kolmogorov and which is called the Fokker–Planck–Kolmogorov (FPK) equation. It is a diffusion-type equation and it is irreversible. After using relations (43) and (44) we get the diffusion equation (46) in the final form:

$$\frac{\partial P(x,t)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial x}\mathscr{D}\frac{\partial P(x,t)}{\partial x} \tag{47}$$

with a *diffusion coefficient*

$$\mathscr{D}=\mathscr{B}=\lim_{\Delta t\to 0}\frac{\langle\langle (\Delta x)^2\rangle\rangle}{\Delta t} \ . \tag{48}$$

Eq. (47) has a divergent form [1] that corresponds to the conservation law of the number of particles:

$$\frac{\partial P}{\partial t} = \frac{\partial J}{\partial x} \tag{49}$$

with the particle flux

$$J = \frac{1}{2}\mathscr{D}\frac{\partial P}{\partial x} \ . \tag{50}$$

In the following section we will see how a similar scheme can be applied to derive the fractional kinetic equation.

An additional condition follows from (44) and notations (45) and (48):

$$\mathscr{A}(x) = \frac{1}{2}\frac{\partial \mathscr{B}(x)}{\partial x} = \frac{1}{2}\frac{\partial \mathscr{D}}{\partial x} \tag{51}$$

which explains a physical meaning of $\mathscr{A}(x)$ as a convective part of the particle flux. This part of the flux and $\mathscr{A}(x)$ are zero if $\mathscr{D} = \text{const}$.

3.3. Solutions and normal transport

There are numerous sources related to the solutions of FPK equation (47) for different initial and boundary conditions (see for example Weiss, 1994; Risken, 1989). Our goal here is just to mention a few simple, important for the future, properties of the FPK equation.

Let us simplify the case considering $\mathscr{D} = \text{const.}$, $x \in (-\infty, \infty)$, and the initial condition for a particle to be at $x = 0$. Then

$$P(x,t) = (2\pi\mathscr{D}t)^{-1/2}\exp\left(-\frac{x^2}{2\mathscr{D}t}\right) \ , \tag{52}$$

known as Gaussian distribution. Its odd moments are zero, second moment is

$$\langle x^2 \rangle = \mathscr{D}t \tag{53}$$

and higher moments are

$$\langle x^{2m} \rangle = \mathscr{D}_m t^m \quad (m = 1, 2, \ldots) \tag{54}$$

with $\mathscr{D}_{m=1} = \mathscr{D}$ and $\mathscr{D}_{m>1}$ can be easily expressed through $\mathscr{D}$ but they do not depend on t. There are two properties that we will refer to in the following section all moments of $P(x,t)$ are finite, as a result of the exponential decay of P for $x \to \infty$, and the distribution $P(x,t)$ is invariant under the renormalization

$$\hat{R}(\lambda)\colon\ x' = \lambda x, \quad t' = \lambda^2 t \tag{55}$$

with arbitrary λ, i.e. the renormalization group $\hat{R}(\lambda)$ is continuous. Evolution of moments $\langle x^m \rangle$ with time will be called *transport*. Dependence (53) and (54) will be called *normal transport*.

[1] The way to derive Eq. (47) is slightly different from what has been used in the original works of Kolmogorov and Landau, due to the use of expansion (39) over the δ-function and its derivatives.

Another type of the distribution function is the so-called moving Gaussian packet

$$P(x,t) = (2\pi\mathscr{D}t)^{-1/2} \exp\left[-\frac{(x-ct)^2}{2\mathscr{D}t}\right] \tag{56}$$

with a velocity c. Distribution (56) satisfies the equation

$$\frac{\partial P}{\partial t} + c\,\frac{\partial P}{\partial x} = \frac{1}{2}\,\mathscr{D}\,\frac{\partial^2 P}{\partial x^2} \tag{57}$$

for which condition (43) or (51) fails. They can be restored if we consider the moments $\langle (x-ct)^m \rangle$. It follows from (56) that

$$\langle x \rangle = ct \tag{58}$$

and

$$\langle (x - \langle x \rangle)^2 \rangle = \mathscr{D}t \tag{59}$$

similarly to (53).

3.4. Kolmogorov conditions and conflict with dynamics

A perfect mathematical scheme often has constraints which limit its application to real phenomena. Constraints related to the Kolmogorov conditions (45) are very important for all problems related to the anomalous transport that will be discussed in the following part of this review. Consider the limit $\delta t \to 0$ and an infinitesimal displacement δx along a particle trajectory that corresponds to this limit. Then $\delta x/\delta t \to v$ where v is the particle velocity, and conditions (45) with notation (48) gives

$$(\delta x)^2/\delta t = v^2 \delta t = \mathscr{D} = \text{const.} \tag{60}$$

This means that v should be infinite in the limit $\delta t \to 0$, which makes no physical sense.

Another manifestation of the conflict can be obtained directly from solution (52) to the FPK equation. This solution satisfies the initial condition

$$P(x,0) = \delta(x) \tag{61}$$

i.e. a particle is at the origin at $t = 0$. For any finite time t solution (52) or (56) has non-zero probability of the particle to be at any arbitrary distant point x, which means the same: the existence of infinite velocities to propagate from $x = 0$ to $x \to \infty$ during an arbitrary small time interval t. A formal acceptance of this result appeals to the exponentially small input from the propagation with infinite velocities. A physical approach to the obstacle in using the FPK equation is to abandon the limit $\Delta t \to 0$ in (45), to introduce $\min \Delta t$, and to consider a limit

$$t/\min \Delta t \to \infty\ . \tag{62}$$

A more serious question is how to use (62) and how the FPK equation can be applied to real dynamics. Let us demonstrate the answer using the standard map (8) as an example.

As it was mentioned in Section 2.2.3, map (8) corresponds to a periodically kicked particle dynamics, i.e. $\min \Delta t = 1$ as it follows from (9). It is known from Chirikov (1979) that for $K \gg 1$

one can consider variable x to be random with almost uniform distribution in the interval $(0, 2\pi)$.[2] Then

$$\langle\langle \sin x \rangle\rangle = 0, \quad \langle\langle \sin^2 x \rangle\rangle = \frac{1}{2} ,$$

$$\Delta p_n \equiv p_{n+1} - p_n, \quad \langle\langle \Delta p_n \rangle\rangle = 0, \quad \langle\langle (\Delta p_n)^2 \rangle\rangle = K^2/2 , \tag{63}$$

where double brackets $\langle\langle \ldots \rangle\rangle$ means averaging over x, and one can write the corresponding FPK equation

$$\frac{\partial P(p,t)}{\partial t} = \frac{1}{2} \mathscr{D}(K) \frac{\partial^2 P(p,t)}{\partial p^2} \tag{64}$$

with

$$\mathscr{D}(K) = K^2/2 . \tag{65}$$

Eq. (64) provides the normal transport. Particularly

$$\langle p^2 \rangle = \frac{1}{2} K^2 t . \tag{66}$$

More sophisticated analysis gives for $\mathscr{D}(K)$ an oscillating behavior

$$\mathscr{D}(K) = K^2 \left(\frac{1}{2} - J_2(K) \right) \tag{67}$$

with $J_2(K)$ as the Bessel function (see Rechester and White, 1980; Rechester et al., 1981), but keeps the same diffusional equation (64). More serious changes to the diffusional equation will be discussed in Section 10. Our main goal here is to show how the described conflict can be eliminated using truncated distributions.

3.5. Truncated distributions

A general scheme to perform a simulation of the problem of diffusion and transport for given dynamical equations is to select a set of initial points in phase space $\{x_0, p_0, t = 0\}$ and let them move until a large time t. Then for different time instants $t_1, t_2, \ldots, t$ one can collect points into boxes located in phase space and create a distribution function $P(x, p, t_j)$ or its projections $P(x, t_j)$, $P(p, t_j)$. All these distributions are always truncated by some values $x_{\max}$, and $p_{\max}$ because velocities of trajectories for all initial conditions are bounded during the finite time interval $(0, t)$. For a fairly large t we can split, for example, $P(p, t)$ into two parts:

$$P(p,t) = P_{\text{core}}(p,t) + P_{\text{tail}}(p,t) \tag{68}$$

and calculate the corresponding moments

$$\langle p^m \rangle = \langle p^m \rangle_{\text{core}} + \langle p^m \rangle_{\text{tail}} . \tag{69}$$

Let us estimate the second term in (69).

[2] Deviations from this approximation will be considered in detail later.

Assume that p^* is the point of splitting of $P(p,t)$ into the core and tail parts. Then

$$\langle p^m \rangle_{\text{tail}} = \int_{p^*}^{p_{\max}} \mathrm{d}p\, p^m P(p,t) < (p^*)^m P(p^*,t)(p_{\max} - p^*) \ . \tag{70}$$

For the Gaussian distribution $P(p^*,t)$ is exponentially small and we can neglect $\langle p^m \rangle_{\text{tail}}$ independently on m. The situation is different if for large values of p the distribution function behaves algebraically, i.e.

$$P(p,t) \sim c(t)/p^{\delta_p} \quad (p \to \infty) \ . \tag{71}$$

All moments $\langle p^m \rangle$ diverge for $m \geqslant \delta_p - 1$ and estimate (70) should be replaced by

$$\langle p^m \rangle_{\text{tail}} = \frac{c(t)}{m - \delta_p + 1}\, p_{\max}^{m-\delta_p+1} \to \infty \quad (p_{\max} \to \infty) \ . \tag{72}$$

Expression (72) shows that for the truncated distribution with an algebraic asymptotics, the time evolution of fairly large moments is defined through the largest value of momentum that a particle can obtain during its dynamics. The result imposes some constraints on how large can m be for the given model with its δ_p and for selected observation time t. Opposite to the Gaussian case, we can neglect $\langle p^m \rangle_{\text{core}}$ in (69).

A similar statement exists for another distribution function $P(x,t)$ if its behavior is algebraic for large $x > 0$, i.e.

$$P(x,t) \sim c(t)/x^{\delta_x} \quad x \to \infty \ . \tag{73}$$

This consideration will be important when we consider anomalous transport in Section 10.

The distribution defined as

$$P(p,t) = \begin{cases} P^{(\text{tr})}(p,t), & 0 < p \leqslant p_{\max} \ , \\ 0, & p > p_{\max} \end{cases} \tag{74}$$

will be called *truncated distribution*, and the corresponding moments will be called *truncated moments*. Any kind of simulations of the direct dynamics deals only with the truncated distributions and moments.

Finally, we arrive at the following important constraints which are necessary for a realistic analysis of the dynamics:

$$\delta t \geqslant \delta t_{\min} \quad (x, p) \leqslant (x_{\max}, p_{\max}) \tag{75}$$

which means the infinitesimal time is bounded from below and the phase space variables are bounded from above. Other consequences of the truncation can be found in Ivanov et al. (2001).

4. Lévy flights and Lévy processes

4.1. General comments

Gaussian distribution (52) links to the large numbers law: the sum of independent random variables is distributed in the same way as any one of them. The uniqueness of the Gaussian distribution with such a property was reconsidered by Lévy (1937) who had formulated a new approach valid for

distributions with an infinite second moment. There is a nice description of the history of Lévy's discovery, as well as its relation to other probabilistic theories and to the St. Petersburg paradox of Daniel Bernoulli in Montroll and Shlesinger (1984). It happened that the Lévy distribution and Lévy processes had a strong impact on different areas of scientific analysis including not only probability theory, but also physics, economics, financial mathematics, geophysics, dynamical systems, etc. In Feller (1949), important theorems were formulated for the limit distributions (see Section 4.3). Mandelbrot (1982) has indicated numerous applications of the Lévy distributions and coined a notion *Lévy flights*. The more contemporary applications were analyzed in Montroll and Shlesinger (1984); Bouchaud and Georges (1990); Geisel (1995), and in a collection of articles by Shlesinger et al. (1995a, b). It became clear that the ideas related to the Lévy processes can be important in the analysis of chaotic dynamics after some modification will be applied (Zaslavsky, 1992, 1994a, b). In this section a very brief information about the Lévy processes will be introduced with a stress on what are the features of the process that can be or cannot be used for the dynamically chaotic trajectories.

4.2. Lévy distribution

Let $P(x)$ be a normalized distribution of a random variable x, i.e.

$$\int_{-\infty}^{\infty} P(x)\,\mathrm{d}x = 1 \tag{76}$$

with a characteristic function

$$P(q) = \int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{e}^{\mathrm{i}qx} P(x)\ . \tag{77}$$

Consider two different random variables x_1 and x_2 and their linear combination

$$cx_3 = c_1x_1 + c_2x_2\ . \tag{78}$$

The law is called stable if all x_1, x_2, x_3 are distributed due to the same function $P(x_j)$. Gaussian distribution

$$P_G(x) = (2\pi\sigma_d)^{-1/2} \exp\left(-\frac{x^2}{2\sigma_d}\right) \tag{79}$$

is an example of the stable distribution with a finite second moment

$$\sigma_d = \langle x^2 \rangle \tag{80}$$

(compare to (52)). Another class of solutions was found by Lévy (1937).

Let us write the equation

$$P(x_3)\,\mathrm{d}x_3 = P(x_1)P(x_2)\delta\left(x_3 - \frac{c_1}{c}x_1 - \frac{c_2}{c}x_2\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\ , \tag{81}$$

where condition (78) has been used. Following definition (77), one can write the equation for characteristic function

$$P(cq) = P(c_1q)P(c_2q) \tag{82}$$

or

$$\ln P(cq) = \ln P(c_1 q) + \ln P(c_2 q) \ . \tag{83}$$

Eqs. (82) and (83) are functional ones with an evident solution

$$\ln P_\alpha(cq) = (cq)^\alpha = c\mathrm{e}^{-\mathrm{i}\alpha \operatorname{sign} q}|q|^\alpha \tag{84}$$

and condition

$$(c_1/c)^\alpha + (c_2/c)^\alpha = 1 \tag{85}$$

which consists of an arbitrary parameter α.

The distribution $P_\alpha(x)$ with the characteristic function

$$P_\alpha(q) = \exp(-c|q|^\alpha) \tag{86}$$

is known as *Lévy distribution* with the *Lévy index* α. For $\alpha=2$ we arrive at the Gaussian distribution $P_G(x)$. An important condition introduced by Lévy is

$$0 < \alpha \leqslant 2 \tag{87}$$

which guarantee positiveness of

$$P_\alpha(x) = \int \mathrm{d}q\, \mathrm{e}^{\mathrm{i}qx} P_\alpha(q) \ . \tag{88}$$

This condition will be discussed later in Section 5.

The case $\alpha = 1$ is known as Cauchy distribution

$$P_1(x) = \frac{c}{\pi} \frac{1}{x^2 + c^2} \ . \tag{89}$$

An important case is the asymptotics of large $|x|$

$$P_\alpha(x) \sim \frac{1}{\pi} \alpha c \Gamma(\alpha) \sin \frac{\pi\alpha}{2} \frac{1}{|x|^{\alpha+1}} \tag{90}$$

(see Lévy, 1937; Feller, 1949, 1957; Uchaikin and Zolotarev, 1999).

4.3. Lévy process

There are many different ways to introduce the Lévy process, i.e. a time-dependent process that at an infinitesimal time has the Lévy distribution of the process variable (Lévy, 1937; Gnedenko and Kolmogorov, 1954; Feller, 1957; Uchaikin and Zolotarev, 1999; Montroll and Shlesinger, 1984). Here we use a simplified version for the infinitely divisible processes.

Consider the transition probability density $P(x_0, t_0; x_N, t_N)$ that satisfies the chain equation

$$P(x_0, t_0; x_N, t_N) = \int \mathrm{d}x_1 \ldots \mathrm{d}x_{N-1} P(x_0, t_0; x_1, t_1) \cdot P(x_1, t_1; x_2, t_2) \ldots P(x_{N-1}, t_{N-1}; x_N, t_N) \tag{91}$$

and put

$$t_{j+1} - t_j = \Delta t, \quad (\forall j); \quad t_N - t_0 = N\Delta t \ . \tag{92}$$

Assume that the process is uniform in time and space, i.e.

$$P(x_j, t_j; x_{j+1}, t_{j+1}) = P(x_{j+1} - x_j; t_{j+1} - t_j) = P(x_{j+1} - x_j; \Delta t) \ . \tag{93}$$

Then (91) transforms into

$$P(x_N - x_0; N\Delta t) = \int \mathrm{d}y_1 \ldots \mathrm{d}y_N P(y_1, \Delta t) \ldots P(y_N; \Delta t) \ , \tag{94}$$

where $y_j = x_j - x_{j-1}$; $j \geqslant 1$.

By introduction the characteristic functions

$$P(q) = \int \mathrm{d}y_j \, \mathrm{e}^{\mathrm{i}qy_j} P(y_j; \Delta t) \quad (j \neq N) \ ,$$

$$P^{(N)}(q) = \int \mathrm{d}y_{(N)} \, \mathrm{e}^{\mathrm{i}qy^{(N)}} P_N(y^{(N)}; N\Delta t), \quad y^{(N)} = \sum_1^N y_j \tag{95}$$

we obtain from (94)

$$P_N(q) = [P(q)]^N \ . \tag{96}$$

Following the concept of stable distributions and using expression (82), let us consider $P(q)$ as a function of two parameters α and c that will be defined later. Namely, change the notation

$$P(q) \to P_\alpha(q; \Delta c), \quad P_N(q) \to P_\alpha(q; c_N) \ , \tag{97}$$

where Δc or c_N should replace c in (86). These equations are consistent if

$$c_N = N\Delta c = N\Delta t \cdot \frac{\Delta c}{\Delta t} \equiv cN\Delta t \tag{98}$$

and Eq. (96) takes the form

$$P_\alpha(q; ct) = \exp(-cN\Delta t|q|^\alpha) \tag{99}$$

with $t = N\Delta t$ and $t_0 = 0$. In the limit $\Delta t \to 0$, $N \to \infty$, $N\Delta t = t$, and $c = \Delta c/\Delta t$ we arrive to the *characteristic function* of the *Lévy process*:

$$P_\alpha(q, t) = \exp(-ct|q|^\alpha) \ . \tag{100}$$

The original Lévy process can be written as the inverse Fourier transform of (100)

$$P_\alpha(x, t) = \int \mathrm{d}q \, \mathrm{e}^{\mathrm{i}qx - ct|q|^\alpha} \tag{101}$$

with the asymptotics for $|x| \to \infty$ similar to (90)

$$P_\alpha(x, t) \sim \frac{1}{\pi} \alpha c \Gamma(\alpha) \sin \frac{\pi\alpha}{2} \cdot \frac{t}{|x|^{\alpha+1}} \ . \tag{102}$$

From (102) we have for the moment of order m (not necessarily integer)

$$\langle |x|^m \rangle = \infty, \quad m \geqslant \alpha \ . \tag{103}$$

The second moment ($m = 2$) is infinite since $\alpha < 2$ (see more about similar derivations in Zaslavsky, 1999; Yanovsky et al., 2000).

4.4. Generalizations

There are different generalizations of Lévy distributions and Lévy processes that can be useful in applications. Mainly they are related to anisotropy of the distributions. For example

$$P_\alpha(q,\xi,c) = \exp\{-c|q|^\alpha[1 + \mathrm{i}\xi \operatorname{sign} q \cdot \tan(\pi\alpha/2)]\}, \quad \alpha \neq 1;\ 0 < \alpha < 2 \tag{104}$$

with

$$\xi = \frac{c^+ - c^-}{c^+ + c^-}, \quad c = a_\alpha(c^+ + c^-)\ , \tag{105}$$

where a_α is defined by the normalization condition and for $\alpha = 1$ $\tan(\pi\alpha/2)$ should be replaced by $(\pi/2)\ln|k|$ (see in Cizeau and Bouchaud, 1994; Yanovsky et al., 2000; Uchaikin, 2000). Distribution (104) provides asymptotics

$$P_\alpha(x,\xi,c) \sim c^\pm/|x|^{\alpha+1}\ . \tag{106}$$

Streaming distribution

$$P_\alpha(x,t) \sim \text{const.}/|x - vt|^{\alpha+1} \tag{107}$$

was considered in Montroll and Shlesinger (1984). Another important generalization, the so-called Lévy walks (Shlesinger et al., 1987) will be considered in Section 6.

4.5. Poincaré recurrences and Feller's theorems

In this section we describe a few results formulated in Feller (1949). They show a connection between the Lévy processes and Poincaré recurrences. Although these results are not related directly to dynamical systems, the Poincaré recurrences distribution plays an important role in kinetics as we will see it in Sections 9–12. From that point, Feller's theorems are a "half-way" to the dynamics.

Consider a small domain A (see Fig. 1) and recurrences to A. The recurrence time is the time interval between two subsequent crossings of the boundary of A by a trajectory of the particle on its way out of A. In the bounded Hamiltonian dynamics the sequence of recurrence times $\{t_j\} \equiv t_1, t_2, \ldots, t_n, \ldots$ is infinite. It is assumed that t_j are mutually independent and they belong to the same class of events with all identical probability distribution function

$$P_{\text{rec}}(\tau) = \text{Prob}\{t_k = \tau\} \quad (\forall k) \tag{108}$$

i.e. independent on k. The integrated probability of recurrences is

$$P_{\text{rec}}^{\text{int}}(t) = \int_0^t \mathrm{d}\tau\, P_{\text{rec}}(\tau) \tag{109}$$

and

$$F(t) = 1 - P_{\text{rec}}^{\text{int}}(t) = \int_t^\infty \mathrm{d}\tau\, P_{\text{rec}}(\tau) \tag{110}$$

has a meaning of a probability that the recurrence time is $\geqslant t$.

Feller (1949) introduced two additional characteristics of the recurrences chain: sum of n recurrence times

$$S_n = t_1 + \cdots + t_n \tag{111}$$

and number of the recurrences N_t during time interval $(0,t)$. An evident connection between them is

$$\mathrm{Prob}\{N_t \geqslant n\} = \mathrm{Prob}\{S_n \leqslant t\} \ . \tag{112}$$

Due to the Kac lemma (17) the mean recurrence time τ_{rec} is finite. Then under the conditions of independence of t_j and finiteness of τ_{rec}, two following theorems are valid (Feller, 1949):

1. If $\sigma^2 < \infty$ then for every fixed ξ

$$\mathrm{Prob}\{S_n - n\tau_{\mathrm{rec}} \leqslant n^{1/2}\sigma\xi\} \to \Phi(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\xi} \mathrm{d}y \exp(-y^2/2)$$

$$\mathrm{Prob}\left\{N_t \geqslant t/\tau_{\mathrm{rec}} - t^{1/2}\frac{\sigma\xi}{\tau_{\mathrm{rec}}^{3/2}}\right\} \to \Phi(\xi) \ , \tag{113}$$

where

$$\sigma^2 = \langle (t_k - \langle t_k \rangle)^2 \rangle = \langle (t_k - \tau_{\mathrm{rec}})^2 \rangle \tag{114}$$

and the connection from (112) has been used

$$n\tau_{\mathrm{rec}} + n^{1/2}\sigma\xi = t \ . \tag{115}$$

A simple meaning of this theorem is that fluctuations of the recurrence time from its mean value τ_{rec} are distributed due to the Gaussian law.

2. Let $F(t)$ in (110) has the asymptotics

$$F(t) = \frac{1}{t^{\alpha}} h(t), \quad 0 < \alpha < 2 \tag{116}$$

with

$$\lim_{x\to\infty} h(cx)/h(x) = 1 \ .$$

Then for $1 < \alpha < 2$

$$\mathrm{Prob}\left\{N_t \geqslant t/\tau_{\mathrm{rec}} - \frac{b_t}{\tau_{\mathrm{rec}}^{(1+\alpha)/\alpha}}\xi\right\} \to P_{\alpha}(\xi) \ , \tag{117}$$

i.e. to the Lévy distribution with $P_{\alpha}(\xi)$ from (90) and b_t to be obtained from the equation

$$F(b_t) \sim 1/t^{1/\alpha} \ .$$

Distribution (117) is *the only possible non-normal distribution for* N_t and $1 < \alpha < 2$.

The last statement of the Feller's theorem does not leave us any possibility to escape the Lévy distribution for physical problems since the condition of the theorem are fairly broad. We do not put the case $0 < \alpha < 1$ since it is forbidden due to the Kac lemma. Indeed, comparing (21), (110) and (116) we obtain $\alpha = \gamma - 1$ and condition (22) means $\alpha > 1$.

The presented theorems extend our information about a universality of distribution of the recurrence times fluctuations as random variables: for a finite dispersion σ^2 they are distributed due to the

Gaussian law, and for an infinite dispersion due to the Lévy law. Nevertheless, analysis of the chaotic dynamical systems does not show this, although the Gaussian or Lévy distributions can sometimes be a good approximation.

4.6. Conflict with dynamics

There are different observations that the recurrences distribution can be of the algebraic type (21) with $\gamma \geqslant 3$ or $\alpha \geqslant 2$ if we apply the notations of the Feller's theorems with a corresponding Lévy index. These observations were obtained by simulation (see, for example, in Zaslavsky and Niyazov, 1997; Zaslavsky et al., 1997; Rakhlin, 2000; Leoncini and Zaslavsky, 2002). Another example, Sinai billiard (Sinai, 1963) with infinite horizon, seems to have $\gamma = 3$ and $\alpha > 2$ up to a logarithmic factor what follows from a qualitative analysis and simulations (Bunimovich and Sinai, 1973; Geisel et al., 1987b; Zacherl et al., 1986; Machta, 1983; Machta and Zwanzig, 1983; Zaslavsky and Edelman, 1997). One can expect different deviations from (117) if we recall that the condition of independency of the recurrence times $\{t_j\}$ in Section 4.5 is a kind of approximation in dynamical systems with chaos. Typically there are correlations between the neighboring steps of Poincaré maps that are important for kinetics. Due to that there is a possibility of deviations from the Lévy distribution and Lévy process, and more general approach is necessary. In other words, the Lévy's idea of distributions with infinite moments can be valid with some restrictions. This issue will be discussed in Section 10.

5. Fractional kinetic equation (FKE)

5.1. General comments

Having in mind a description of chaotic trajectories, we can consider first a situation when the complexity (Badii and Politi, 1997) of the trajectories have a hierarchical structure. For example, a trajectory can be presented with different levels of accuracy, or different levels of coarse-graining ΔS in phase space and Δt in time. The more complicated the details, the higher the level of complexity. While trajectories can be arbitrary smooth, their approximation can be a singular one and may require correctly defined conditions of the applicability. The hierarchical sequence of curves can be continued up to some level n after which the property of occurrence of more and more smaller details stops due to physical reasons. Nevertheless we can consider a formal limit $n \to \infty$ of the hierarchical complexity, and arrive in that way to a singular fractal representation of the trajectories which will be valid up to the sizes $\Delta\ell < \Delta\ell_0$. The gain of considering singular trajectories instead of the smooth ones is in a simplification of the theory by introducing some invariant features of dynamics, namely renormalization group approach. This section consists of such a type of approach to the kinetics of chaotic dynamics, based on the results of Zaslavsky (1992, 1994a, b). The derived equation is called Fractional Fokker–Planck–Kolmogorov Equation (FFPK) or simply fractional kinetic equation (FKE). We also will use the notion of *fractional kinetics* (FK).

The idea of exploiting fractional calculus and presenting kinetics in a form of an equation with fractional integro-differentiation is not new. For example some variants of FK were used in Mandelbrot and Van Ness (1968) for signals; Young et al. (1989) for kinetics of advected particles; Isichenko

(1992) and Milovanov (2001) for the problem of percolation; Hanson et al. (1985) for kinetics through cantori for the standard map; Nigmatullin (1986) for the porous media; Douglas et al. (1986, 1987) in macromolecules (see also a review paper by Douglas, 2000); Hilfer (1993, 1995a, b) for evolution and thermodynamics; West and Grigolini (2000) for time series. We will also present here some ideas for the solution of FKE based on Saichev and Zaslavsky (1997) as more motivated for the dynamical systems analysis. Additional material can be found in Hilber (2000) and in Metzler and Klafter (2000). The necessary material on fractional calculus is in the appendices.

5.2. Derivation of FKE

Following Zaslavsky (1992, 1994a, b), we provide in this section a simplified version of the FKE. Let us assume that the transition probability $W(x,x_0;t+\Delta t)$ with an infinitesimal Δt has a new expansion, instead of (34) and (35):

$$\lim_{\Delta t\to 0}\frac{1}{|\Delta t|^{\beta}}\{W(x,x_0;t+\Delta t)-W(x,x_0,t)\}$$
$$=\frac{\partial^{\beta}W(x,x_0;t)}{\partial t^{\beta}}=\frac{\partial^{\beta}P(x,t)}{\partial t^{\beta}}\quad(0<\beta\leqslant 1,\ t\geqslant 0)\ , \tag{118}$$

where we use notation (36) and properties of fractional derivatives (see Appendix A). We also introduce the expansion for $W(x,y;\Delta t)$ instead of (39):

$$W(x,y;\Delta t)=\delta(x-y)+A(y;\Delta t)\delta^{(\alpha)}(x-y)+B(y;\Delta t)\delta^{(\alpha_1)}(x-y)\quad(0<\alpha<\alpha_1\leqslant 2) \tag{119}$$

with appropriate fractal dimension characteristics α and α_1. It is still possible to write for the coefficient $B(y;\Delta t)$ its definition through a moment of W:

$$\langle\langle|\Delta x|^{\alpha_1}\rangle\rangle\equiv\int \mathrm{d}x\,|x-y|^{\alpha_1}W(x,y;\Delta t)=\Gamma(1+\alpha_1)B(y;\Delta t) \tag{120}$$

which is similar to (42) but the coefficient $A(y;t)$ does not have so simple interpretation for the general case unless $B=0$.

By integrating (119) over y we obtain a relation

$$\frac{\partial^{\alpha}\mathscr{A}(x)}{\partial(-x)^{\alpha}}+\frac{\partial^{\alpha_1}\mathscr{B}(x)}{\partial(-x)^{\alpha_1}}=0\ , \tag{121}$$

where

$$\mathscr{A}(x)=\lim_{\Delta t\to 0}\frac{A(x,\Delta t)}{(\Delta t)^{\beta}}$$
$$\mathscr{B}(x)=\lim_{\Delta t\to 0}\frac{B(y;\Delta t)}{(\Delta t)^{\beta}}=\frac{1}{\Gamma(1+\alpha_1)}\lim_{\Delta t\to 0}\frac{\langle\langle|\Delta x|^{\alpha_1}\rangle\rangle}{(\Delta t)^{\beta}} \tag{122}$$

similar to (35). Eq. (121) is a generalization of (43) for the detailed balance principle. Existence of limits (122) when $\Delta t\to 0$ is instead of the Kolmogorov conditions (45). Fractional values of α, α_1 and β represents a new type of the fractal properties of the coarse-grained dynamics.

A particular case is $\alpha_1 = \alpha + 1$, and (121) transforms into

$$\frac{\partial^\alpha}{\partial(-x)^\alpha}\left[\mathscr{A}(x) - \frac{\partial\mathscr{B}(x)}{\partial x}\right] = 0 \tag{123}$$

equivalent to (51) with

$$\mathscr{B}(x) = \frac{1}{\Gamma(2+\alpha)} \lim_{\Delta t\to 0} \frac{\langle\langle|\Delta x|^{\alpha+1}\rangle\rangle}{(\Delta t)^\beta} ,$$

$$\mathscr{A}(x) = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t\to 0} \frac{\langle\langle|\Delta x|^{\alpha}\rangle\rangle}{(\Delta t)^\beta} \tag{124}$$

and the generalization of the Landau formulas (43) and (44):

$$\Delta t \to 0\colon \frac{\langle\langle|\Delta x|^{\alpha}\rangle\rangle}{(\Delta t)^\beta} = \frac{\Gamma(1+\alpha)}{\Gamma(2+\alpha)} \frac{\partial}{\partial x} \frac{\langle\langle|\Delta x|^{\alpha+1}\rangle\rangle}{(\Delta t)^\beta} . \tag{125}$$

The existence of limits (124) can be considered as generalized Kolmogorov condition (compare to (45)).

The FKE can be derived from (118) rewritten on the basis of (32):

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^\beta}\left\{\int \mathrm{d}y\,[W(x,y;t+\Delta t) - \delta(x-y)]P(y,t)\right\} \tag{126}$$

or using expansion (119) and definitions (122)

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial(-x)^\alpha}(\mathscr{A}(x)P(x,y)) + \frac{\partial^{\alpha_1}}{\partial(-x)^{\alpha_1}}(\mathscr{B}(x)P(x,y)) . \tag{127}$$

This equation can be simplified in the case $\alpha_1 = \alpha + 1$

$$\frac{\partial^\beta P}{\partial t^\beta} = -\frac{\partial^\alpha}{\partial(-x)^\alpha}\left(\mathscr{B}\frac{\partial P}{\partial x}\right) \tag{128}$$

which transfers into regular diffusion equation for $\alpha = 1$ and $\mathscr{B} = \frac{1}{2}\mathscr{D}$.

Fractional derivatives are well defined in a specified direction (see Appendix A), and there is no simple replacement $x \to -x$ or $t \to -t$. That is why a more general operator should be considered, for example, instead of the derivative of order α one can consider

$$\hat{L}_x^{(\alpha)} = \mathscr{A}^+\partial^\alpha/\partial x^\alpha + \mathscr{A}^-\partial^\alpha/\partial(-x)^\alpha . \tag{129}$$

For a symmetric case one can use Riesz derivative

$$\frac{\partial^\alpha}{\partial|x|^\alpha} = -\frac{1}{2\cos(\pi\alpha/2)}\left[\frac{\partial^\alpha}{\partial x^\alpha} + \frac{\partial^\alpha}{\partial(-x)^\alpha}\right] \quad (\alpha \neq 1) . \tag{130}$$

The corresponding FKE (127) takes the form (Saichev and Zaslavsky, 1997)

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^\alpha}{\partial|x|^\alpha}(\mathscr{A}P) + \frac{\partial^{\alpha_1}}{\partial|x|^{\alpha_1}}(\mathscr{B}P), \quad 0 < \alpha < \alpha_1 \leqslant 2 . \tag{131}$$

In the case, when the term with $\mathscr{B}$ can be neglected, we have a simplified version of FKE

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^\alpha}{\partial |x|^\alpha}(\mathscr{A}P) \ . \tag{132}$$

In the case $\beta = 1$, $\alpha = 2$ it is a normal diffusion equation. For $0 < \beta < 1$, $\alpha = 2$

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^2}{\partial x^2}(\mathscr{A}P) \quad (\beta < 1) \tag{133}$$

it is called the equation of fractional Brownian motion (Mandelbrot and Van Ness, 1968; Montroll and Shlesinger, 1984). For $\beta = 1$ and $1 < \alpha < 2$ the FKE corresponds to the Lévy process (see Section 4.3):

$$\frac{\partial P}{\partial t} = \frac{\partial^\alpha}{\partial |x|^\alpha}(\mathscr{A}P) \quad (1 < \alpha < 2) \ . \tag{134}$$

There are other different forms for the FKE that generalize forms (131) and (134): anisotropic equations were considered in Yanovsky et al. (2000); Meerschaert et al. (2001); directioned equations (Weitzner and Zaslavsky, 2001; Meerschaert et al., 2001); non-linear fractional equations (Biler et al., 1998; Barkai, 2001; Schertzer et al., 2001). Some of these equations will be considered later.

We should mention that the investigation of the type and properties of the FKE is at its beginning stage and a number of important questions are not answered yet. Some of these questions will be discussed in the following sections.

Parameters β, α, α_1 will be called the *critical exponents*.

5.3. Conditions for the solutions to FKE

Any type of the FKE and its generalization can be considered as independent mathematical problems. If we want to stay closely to the specific applications of the FKE to dynamical systems, restrictions related to the physical nature and the origin of the FKE should be imposed. Before considering solutions to the FKE, let us make a few comments about some constraints. Other conditions will be done later.

(a) *Interval condition.* We should define interval of consideration in space–time. Speaking about the space, we have in mind phase space (coordinate–momentum) and the variable x can represent any of them. The infinite intervals assume a possibility to have infinite moments of $P(x,t)$ while finite intervals $(x_{\min}, x_{\max})$, $(t_{\min}, t_{\max})$, that will be called *space* and *time windows*, lead to the all finite moments since $P(x,t)$ is integrable.

(b) *Positiveness.* Solution $P(x,t)$ has the meaning of probability, and it should be positively defined, i.e.

$$P(x,t) \geqslant 0 \tag{135}$$

in the domain of consideration. For the infinite space–time domains condition (135) leads to restrictions on the possible values of critical exponents. Particularly we have the condition $0 < \alpha \leqslant 2$ for the Lévy processes ($\beta = 1$) or the condition:

$$0 < \beta \leqslant 1, \quad 0 < \alpha \leqslant 2 \tag{136}$$

(Saichev and Zaslavsky, 1997) for (132) with $\mathscr{A} = \text{const}$. A rigorous consideration of the $P(x,t)$ positiveness for both fractal values of (α, β) does not exist yet. Examples of the violation of (136) will be shown later.

(c) *Intermediate asymptotics.* It can be that dynamics imposes different asymptotics for different space–time windows. An example of particles advection in convective flow can be found in Young et al. (1989). Other examples will be indicated later. For such cases should be different pairs (α_j, β_j) for different windows.

(d) *Evolution of moments.* It is easy to obtain the time-dependence of some moments if $\mathscr{A}=\text{const.}$ in (132). Let us multiply the equation by $|x|^\alpha$ and integrate it over x. Then

$$\begin{aligned}\frac{\partial^\beta \langle |x|^\alpha \rangle}{\partial t^\beta} &= \mathscr{A} \int \mathrm{d}x\, |x|^\alpha \frac{\partial P(x,t)}{\partial |x|^\alpha} \\ &= \mathscr{A} \int \mathrm{d}x\, P(x,t) \frac{\partial^\alpha}{\partial |x|^\alpha} |x|^\alpha = \mathscr{A}\Gamma(1+\alpha) \ , \end{aligned} \tag{137}$$

where we introduce the moment of $P(x,t)$

$$\langle |x|^\alpha \rangle = \int \mathrm{d}x\, |x|^\alpha P(x,t) \tag{138}$$

and use the formulas from Appendix A. After integrating (137) over t^β (or differentiation with respect to $t^{-\beta}$) we obtain

$$\langle |x|^\alpha \rangle = \mathscr{A} \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)} t^\beta \ . \tag{139}$$

For the case of the self-similarity of the solution for FKE, which we will discuss more in Section 5.4, one may expect

$$\langle |x| \rangle \sim t^{\beta/\alpha} = t^{\mu/2} \ , \tag{140}$$

where we introduce the *transport exponent*

$$\mu \equiv 2\beta/\alpha \ . \tag{141}$$

(e) *Definition of fractional integro-differentiation.* This definition is not unique (Samko et al., 1987) and there is a discussion of using different definitions for space and time derivatives (Chukbar, 1995). In fact, one can use any integral transform and a type of the chosen derivative will influence boundary-initial conditions and the way how the solution $P(x,t)$ will be constructed by the inverse transform. More accurately, one can say that the distribution $P(x,t)$ is defined not only by the boundary-initial conditions but also by the type of derivatives used in FKE. A similar situation exists with Fourier or Laplace transforms. Any of them can be used if we know how to select a contour of integration in the complex plane to satisfy the boundary-initial conditions.

5.4. Solutions to FKE (series)

In this section we consider FKE in the form

$$\frac{\partial^\beta}{\partial t^\beta} P(x,t) = \frac{\partial^\alpha}{\partial |x|^\alpha} P(x,t) + S(x,t) \ , \tag{142}$$

where $S(x,t)$ is a source. For the source we use the standard form

$$S(x,t) = \delta(x)\delta(t) \tag{143}$$

or a time distributed point source

$$S(x,t) = S_0(t)\delta(x) \ ,$$

$$S_0(t) = t^{-\beta}/\Gamma(1-\beta) = \frac{\partial^\beta}{\partial t^\beta}\,\hat{1} \to_{\beta\to 1} \delta(t) \tag{144}$$

which will be explained later and where

$$\hat{1} = \begin{cases} 1, & t > 0 \ , \\ 0, & t < 0 \ . \end{cases}$$

There are two ways to proceed: using infinite series (Saichev and Woyczynski, 1997; Saichev and Zaslavsky, 1997; Zolotarev et al., 1999), or using Fourier–Laplace transform (Afanasiev et al., 1991; Fogedby, 1994; Compte, 1996; Meerschaert et al., 2001; Weitzner and Zaslavsky, 2001).

Consider a Fourier transform with respect to x

$$P(q,t) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx} P(x,t) \tag{145}$$

and similarly for source (144)

$$S(q,t) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx} S(x,t) = S_0(t) \ . \tag{146}$$

Eq. (142) transforms into

$$\frac{\partial^\beta}{\partial t^\beta} P(q,t) + |q|^\alpha P(q,t) = S_0(t) \quad (t > 0) \ . \tag{147}$$

For case (146) solution for (147) can be written in the form (Saichev and Zaslavsky, 1997)

$$\begin{aligned} P(q,t) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m\beta+1)} (|q|^\alpha t^\beta)^m \\ &= E_\beta(-|q|^\alpha t^\beta) \quad (t > 0) \ , \end{aligned} \tag{148}$$

where

$$E_\beta(z) = \sum_{m=0}^{\infty} z^m/\Gamma(m\beta+1) \tag{149}$$

is the Mittag–Leffler function of order β (see Appendix C).

Using the inverse transform

$$P(x,t) = \int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{e}^{-\mathrm{i}qx} P(q,t) \tag{150}$$

we can write the solution in the series form

$$P(x,t) = \frac{1}{\pi|y|}\,\frac{1}{t^{\mu/2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{|y|^{m\alpha}}\,\frac{\Gamma(m\alpha+1)}{\Gamma(m\beta+1)} \cos\left[\frac{\pi}{2}(m\alpha+1)\right] \ , \tag{151}$$

where

$$y = x/t^{\mu/2} \tag{152}$$

and μ is the same as in (141). The asymptotics of (151) is

$$P(x,t) \sim \frac{1}{\pi}\frac{t^{\beta}}{|x|^{\alpha+1}}\frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\sin\frac{\pi\alpha}{2} \tag{153}$$

under condition

$$|x|^2 \gg t^{\mu} \ . \tag{154}$$

Expression (153) coincides for $\beta = 1$ with the Lévy process distribution $P_\alpha(x,t)$ in (102). We can conclude from (153) that the moments

$$\langle |x|^{\delta}\rangle < \infty \quad \text{if } 0 < \delta < \alpha \leqslant 2 \ . \tag{155}$$

Series (148) also shows that the solution can be considered in a self-similar form

$$P(x,t) = t^{-\mu/2}P_{(\alpha,\beta)}(|x|/t^{-\mu/2}) \tag{156}$$

which we will discuss later. For $\alpha = 2$, $\beta \leqslant 1$ all moments $\langle x^m\rangle$ are finite.

5.5. Solutions to FKE (separation of variables)

This section having more technical details, provides an important physical interpretation of the FKE (Saichev and Zaslavsky, 1997).

Consider Laplace in time and Fourier in space transforms of $P(x,t)$:

$$P(q,u) = \int_0^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ut+\mathrm{i}qx}P(x,t) \tag{157}$$

with the inverse formula

$$P(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{e}^{ut-\mathrm{i}qx}P(x,t) \ . \tag{158}$$

Application of (157) to the FKE (142) with source (143) gives

$$P_{\alpha,\beta}(q,u) = 1/(u^{\beta} + |q|^{\alpha}) \tag{159}$$

(see Appendix B for integral transforms of fractional derivatives). This result can be rewritten in the form

$$P_{\alpha,\beta}(q,u) = \int_0^{\infty} \mathrm{d}s \exp[-s(u^{\beta} + |q|^{\alpha})] = \int_0^{\infty} \mathrm{d}s\, Q_{\beta}(u,s)W_{\alpha}(q,s) \tag{160}$$

with

$$Q_{\beta}(u,s) = \mathrm{e}^{-su^{\beta}}, \quad W_{\alpha}(q,s) = \mathrm{e}^{-s|q|^{\alpha}} \ . \tag{161}$$

$Q_{\beta}(u,s)$ and $W_{\alpha}(q,s)$ can be interpreted as corresponding Lévy-type characteristic functions with a parameter s, and (160) is separation of variables formulae for the characteristic function $P_{\alpha,\beta}(q,u)$ of the solution.

Let us apply now the inverse transform (158) and use presentation (160):

$$P_{\alpha,\beta}(x,t) = \int_0^\infty \mathrm{d}s Q_\beta(t,s) W_\alpha(x,s) \tag{162}$$

with

$$Q_\beta(t,s) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}u\, \mathrm{e}^{ut} Q_\beta(u,s) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}u \exp(ut - su^\beta) \ ,$$

$$W_\alpha(x,s) = \int_{-\infty}^{\infty} \mathrm{d}q\, \mathrm{e}^{-\mathrm{i}qx} W_\alpha(q,s) = \int_{-\infty}^{\infty} \mathrm{d}q \exp(-\mathrm{i}qx - s|q|^\alpha) \ . \tag{163}$$

Result (162) completes the separation of variables and presents the solution through a superposition of Lévy processes along x and t. A similar presentation can be developed for the more complicated source (144) (see also Afanasiev et al., 1991; Fogedby, 1994; Compte, 1996).

5.6. Conflict with dynamics

The limitations for application of the FKE to dynamical systems can be compared to the limitations for diffusional process described in Section 4.6. Consider analogs (124) and (125) to the Kolmogorov condition (45):

$$(\delta x)^\alpha/(\delta t)^\beta = v^\alpha (\delta t)^{\alpha-\beta} = \mathscr{A} = \text{const.} \quad (\delta t \to 0) \ . \tag{164}$$

At the same time,

$$\alpha - \beta = \alpha(1 - \beta/\alpha) = \alpha(1 - \mu/2) > 0 \tag{165}$$

since $\mu < 2$. This means that in the limit $\delta t \to 0$ should be $v \to \infty$, and we arrive at the same conflict as in the normal diffusion case. The resolution of this conflict is similar to the case of normal diffusion: there exists $\delta t_{\min}$ such that for $\delta t < \delta t_{\min}$ the FKE cannot be applied.

More serious constraints are imposed by the condition of positiveness of $P_{\alpha,\beta}(x,t)$, i.e. $\beta < 1$, $0 < \alpha < 2$. Some simulations show, as we will see later, the values $\alpha > 2$. A theory of the FKE is not developed yet for $\beta > 1$ and $\alpha > 2$.

We also need to consider truncated distribution function

$$P_{\alpha,\beta}(x,t) = \begin{cases} P^{(\mathrm{tr})}_{\alpha,\beta}(x,t), & 0 < x \leqslant x_{\max} \ , \\ 0, & x > x_{\max} \end{cases} \tag{166}$$

(compare to (3.5.7)) and truncated moments

$$\langle |x|^m \rangle_{\mathrm{tr}} = \int \mathrm{d}x\, P^{(\mathrm{tr})}_{\alpha,\beta}(x,t) < \infty \tag{167}$$

in order to avoid infinite velocities in the solutions, forbidden by the dynamics. All these comments will be necessary when real experimental or simulation data are compared to the theory. As it follows from (156)

$$\langle |x|^m \rangle_{\mathrm{tr}} \sim t^{m\mu/2} \tag{168}$$

and all truncated moments are finite, although there is a restriction on the value of m which depends on $t_{\max}$ (see the discussion in Section 3.5). Particularly

$$\langle |x|^2 \rangle_{\mathrm{tr}} \sim t^{\mu} \tag{169}$$

introducing the transport exponent μ.

For $\beta = 1$ we have the Lévy process with $\mu = 2/\alpha < 2$. This means that the permitted values of α are

$$1 < \alpha < 2 \tag{170}$$

and it is not clear from the dynamics why the values of $0 < \alpha < 1$ are not achievable. For the interval (170) $\mu > 1$, i.e. the transport is superdiffusive, and it is not clear from the dynamics if subdiffusive transport with $\mu < 1$ is forbidden or not, while the subdiffusion was observed in Schwägerl and Krug (1991) and in Afanasiev et al. (1991). The value $\mu = 2$ corresponds to the pure ballistic case. This will be discussed more in Section 9.5.

6. Probabilistic models of fractional kinetics

6.1. General comments

We use the notion "probabilistic models" to underline the fact that the main axiomatic approach to the final kinetic equation is based on the construction of random wandering process rather than on the construction of a model for dynamic trajectories. The most popular and widely used is the so-called *Continuous Time Random Walk* (CTRW) model introduced in Montroll and Weiss (1965) and developed in a number of succeeding publications (Weiss 1994; Montroll and Shlesinger, 1984; Shlesinger et al., 1987; Uchaikin, 2000; Metzler and Klafter, 2001). Some other models also will be considered in this section and their relations to each other will be discussed.

6.2. Continuous time random walk (CTRW)

CTRW theory operates with two kinds of probabilities related to the random walk of a particle: probability density to find a particle at the position x after executing j steps $P_j(x)$, and probability density to find a particle at x at time $t, P(x,t)$. The corresponding random walk equations assume independence of changes at each step together with a uniformity of the probability distributions of time and distance intervals between consequent steps. Under these conditions we have for $P_j(x)$

$$P_{j+1}(x) = \int_{-\infty}^{\infty} \mathrm{d}x' W(x - x') P_j(x') \tag{171}$$

with $W(x)$ as a transitional probability, and

$$P(x,t) = \int_0^t \mathrm{d}\tau\, \phi(t - \tau) Q(x,\tau) \tag{172}$$

with $\phi(t)$ as a probability density to stay in the achieved position x for a time t, and $Q(x,t)$ as a probability to find a particle in x at t immediately after a step has been taken.

The probability $\phi(t)$ can be expressed through the probability density $\psi(t)$ to have an interval t between two consequent steps:

$$\phi(t) = \int_t^\infty \mathrm{d}t' \psi(t') \ . \tag{173}$$

Let $\psi_j(t)$ be the probability density to make the jth step at a time instant t. Then

$$\psi_{j+1}(t) = \int_0^t \mathrm{d}\tau\, \psi(t-\tau)\psi_j(\tau), \quad \psi_1(t) \equiv \psi(t) \ . \tag{174}$$

The following obvious equation:

$$Q(x,t) = \sum_{j=1}^\infty \psi_j(t) P_j(x) \tag{175}$$

establishes the connection between discrete steps random walk and continues waiting time at different steps.

It is convenient to use Fourier transforms of $P_j(x)$ and $W(x)$:

$$P_j(q) = \frac{1}{(2\pi)} \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx} P_j(x) \ ,$$

$$W_j(q) = \frac{1}{(2\pi)} \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{i}qx} W(x) \tag{176}$$

which for simplicity are written in one-dimensional case. From (171) and (176) we obtain

$$P_j(q) = W(q) P_{j-1}(q) = [W(q)]^j \tag{177}$$

with an initial condition $P_0(q) = 1$, i.e. the particle is initially at the origin. Thus, from (177)

$$P_j(x) = \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}qx} [W(q)]^j \ . \tag{178}$$

Consider Laplace transform of $\psi_j(t), \psi(t)$ and $Q(x,t)$:

$$\psi_j(u) = \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-ut} \psi_j(t) \ ,$$

$$\psi(u) = \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-ut} \psi(t) \ , \tag{179}$$

$$Q(x,u) = \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-ut} Q(x,t) \ .$$

Then it follows from (174) and (179)

$$\psi_j(u) = \psi(u)\psi_{j-1}(u) = [\psi(u)]^j \ . \tag{180}$$

After combining (172), (179) with (175) we obtain

$$Q(x,u)=\int_{-\infty}^{\infty}\mathrm{d}x\frac{\mathrm{e}^{-\mathrm{i}qx}}{1-W(q)\psi(u)}\ . \tag{181}$$

Finally, using connection (173) in the form

$$\phi(u)=[1-\psi(u)]/u \tag{182}$$

and applying it and (181) to (172) we arrive at the solution:

$$P(x,t)=\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{c+\infty}\mathrm{d}u\,\frac{1}{u}\,[1-\psi(u)]\mathrm{e}^{ut}\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}q\,\frac{\mathrm{e}^{-\mathrm{i}qx}}{1-\psi(u)W(q)} \tag{183}$$

which the *Montroll–Weiss equation* (*MWE*). The final result has been expressed through the two basic probabilities: $W(x)$ for the length of a step, and $\psi(t)$ for the time interval between the consequent steps.

There are numerous investigations of the MWE (Weiss, 1994; Montroll and Shlesinger, 1984; Metzler and Klafter, 2000), as well as its generalization (Shlesinger et al., 1987; Uchaikin, 2000).

6.3. Asymptotics for CTRW

The main applications of the MWE are related to the limits $t\to\infty$, $x\to\infty$ and corresponding approximations with two transition probability functions $\psi(u)$ and $W(q)$ which forms a considerable model of the random walk process. By choosing $\psi(u)$ and $W(q)$ one can create a model that presumably should fit the physical situation. Eq. (183) is just a formal presentation of the solutions that can be very different depending on the model functions $W(q)$ and $\psi(u)$. Let us rewrite (183) in the form

$$[1-\psi(u)W(q)]P(q,u)=\frac{1}{u}\,[1-\psi(u)] \tag{184}$$

and consider some cases for $\psi(u)$ and $W(q)$. For example we can assume that

$$\psi(u)=1+t_0u,\qquad W(q)=1+x_0q-\frac{1}{2}\,\sigma q^2 \tag{185}$$

considering the limits u, $q\to 0$ equivalent to the limits $t\to\infty$, $x\to\infty$, and introducing constants t_0, x_0, and σ. Then, up to the higher orders terms, (184) becomes

$$\left(t_0u+x_0q-\frac{1}{2}\,\sigma q^2\right)P(q,u)=t_0 \tag{186}$$

which is equivalent to the diffusion equation

$$t_0\,\frac{\partial P(x,t)}{\partial t}+x_0\,\frac{\partial P(x,t)}{\partial x}=\frac{1}{2}\,\sigma\,\frac{\partial^2P(x,t)}{\partial x^2}+t_0\delta(x)\delta(t)\ . \tag{187}$$

It is also assumed that t_0, $\sigma\neq 0$.

Consider now the case

$$W(q)=1-|q|^{\alpha},\ \ \psi(u)=1-u^{\beta}\quad(u,q\to 0) \tag{188}$$

(Afanasiev et al., 1991; Weiss, 1994; Uchaikin, 2000), where constants are included into the definition of q and u. Substitution of (188) into (184) gives

$$(u^\beta + |q|^\alpha)P(q,u) = u^{\beta-1} \ , \tag{189}$$

where we neglect the product $u^\beta|q|^\alpha$, or

$$P(q,u) = \frac{u^{\beta-1}}{u^\beta + |q|^\alpha} \tag{190}$$

and

$$P(x,t) = \frac{1}{2\pi}\,\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}u\int_{-\infty}^{\infty}\mathrm{d}q\frac{u^{\beta-1}}{u^\beta + |q|^\alpha} \ . \tag{191}$$

The last expression can be used to obtain the moments $\langle |x|^\delta\rangle$. By replacing the variables we can obtain

$$\langle |x|^\delta\rangle = \text{const. } t^{\beta/\alpha} = \text{const. } \cdot t^{\mu/2} \tag{192}$$

(Afanasiev et al., 1991). This result coincides with (140) for the FKE. In fact, it is a consequence of the Eq. (189) that coincides with (142) if the source will be taken in form (144). To prove this, let us use the formulas of the Laplace–Fourier transforms of the derivatives (see the Appendix B):

$$\begin{aligned}
&\frac{\mathrm{d}^\alpha}{\mathrm{d}|x|^\alpha}g(x) \overset{(F)}{\to} -|q|^\alpha g(q) \ , \\
&\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}g(t) \overset{(L)}{\to} u^\beta g(u) \ , \\
&\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\hat{1} \overset{(L)}{\to} u^{\beta-1} \ ,
\end{aligned} \tag{193}$$

where (F) and (L) indicates Fourier and Laplace transforms. Applying (193) and (144) to Eq. (189), we arrive at

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha P(x,t)}{\partial |x|^\alpha} + \frac{t^\beta}{\Gamma(1-\beta)}\,\delta(x) \tag{194}$$

which coincides with the FKE (140) when the source is taken in the special form (144). This result establishes the connection between the FKE and CTRW as well as differences since the equations are obtained in different approximations and different initial assumptions. For example the MWE equation (184) or (183) is fairly general and does not use approximations $q \to 0$, $u \to 0$ while the transition to FKE does. At the same time, in the limit $q \to 0$, $u \to 0$ MWE has a special type of source (144) while the FKE does not specify the source. In addition, the FKE in form (194) or (190) can be valid not only in the limit $q \to 0$, $u \to 0$ but for fairly small time–space scales as we will see for dynamical systems.

6.4. Generalizations of CTRW

There are few generalizations that enhance possible applications of the CTRW theory: they are Lévy walk (Shlesinger et al., 1987), relativistic model of CTRW (Shlesinger et al., 1995a, b),

multiplicative random walks (Shlesinger and Klafter, 1989), and others. Here we would like to stop briefly on the so-called Lévy walk which was introduced in Shlesinger et al. (1987) in order to avoid infinite moments of the displacement x. In the same scheme of derivation of the CTRW, let a probability density $\psi(x,t)$ to make a jump of the length x in a time t is

$$\psi(x,t) = \psi(t|x)W(x) \ , \tag{195}$$

where $W(x)$ has the same meaning as in (171) and $\psi(x|t)$ is a conditional probability. It is also possible to write

$$\psi(x,t) = W(x|t)\psi(t) \tag{196}$$

with $\psi(t)$ the same as in (173) and $W(x|t)$ as a conditional probability.

As a simple demonstration, consider an expression

$$\psi(t|x) = \delta(t - x/V(x)) \ , \tag{197}$$

where $V(x)$ is a velocity as a function or the displacement. This form appeared in Shlesinger and Klafter (1989) to show that for the Kolmogorov scaling $V(x) \sim x^{1/3}$ and $W(x) \sim |x|^{1+\beta}$ one could find for the mean square displacement $x(t)$:

$$\langle x(t)^2 \rangle = \begin{cases} t^3, & \beta \leqslant 1/3 \ , \\ t^{2+(3/2)(1-\beta)}, & 1/3 \leqslant \beta \leqslant 1/2 \ , \\ t, & \beta \geqslant 1/2 \ . \end{cases} \tag{198}$$

The idea of the Lévy walk is to couple space-time memory by using expressions (195)–(197) and to obtain, in this way, the finite second moment of the displacement at time instant t while the distribution of all possible displacements $W(x)$ has infinite second moment (see more discussions of that in Section 5.6).

One additional type of generalization of the CTRW can be useful for applications. This is related to the so-called master equation. It was proposed in Montroll and Shlesinger (1984) to write the master equation in the form

$$\frac{\partial P(x,t)}{\partial t} = \int_0^t \mathrm{d}\tau\, \phi(t-\tau) \left\{ \sum_{x'} W(x-x')P(x',t) - P(x,\tau) \right\} \tag{199}$$

with an appropriate transition probability $W(x)$ and a delay probability function $\phi(t-\tau)$. Following (182) and (188), we can assume that $\phi(u) \sim u^\beta$ or $\phi(t) \sim 1/t^{1+\beta}$.

A generalized fractional Kolmogorov–Feller equation was proposed in Saichev and Zaslavsky (1997):

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \int_{-\infty}^{\infty} \mathrm{d}y\, W(t)[P(x-y;t) - P(x,t)] \tag{200}$$

which is similar to (199).

6.5. Conflict with dynamics

The CTRW or MWE equation are governed by two arbitrary probabilistic functions $W(x)$ and $\psi(t)$, and there are high resources for applications when these probabilities appear from known processes. Chaotic dynamics, in its contemporary stage, does not provide information about $W(x)$ and $\psi(t)$ except for maybe some special solutions. In lieu of this comment, one should be very careful in applying the CTRW to experimental and simulation data. The best example is related to the basic assumption for the derivation of the CTRW equation: independence of steps in the random walk (see Section 6.2), which definitely is not the case for the standard map, web map, and all other more complicated models of chaos. A similar comment can be made for the Lévy walk. The dynamic equations and, for example, the standard map do not impose the relation of (195)–(197) type and the observations of many other dynamical models show an absence of the strong coupling of (197) type (see for example in Kuznetsov and Zaslavsky, 2000).

7. More fractional equations

Literature related to the equations with fractional derivatives is fairly vast. Although it is not a subject of this review, we would like to mention some publications that can be important in the study of dynamics: general approach to the media with a fractal support, such as the porous media (Nigmatullin, 1986); fractional wave equation (Wyss, 1986; Schneider and Wyss, 1989; Mainardi, 1996a, b); different equations including fractional differences (Mainardi, 1997; Gorenflo et al., 1999); stable distributions and related equations (Uchaikin, 2000; Uchaikin and Zolotarev, 1999); asymmetric generalization of the FK (Yanovsky et al., 2000). In this section we will consider only a few different cases that seems to be important for applications to the dynamics.

7.1. Weierstrass random walk (WRW)

The Weierstrass random walk (WRW) was coined in Hughes et al. (1981) (see also Montroll and Shlesinger, 1984). Although the WRW is not as well known as CTRW and it is not as convenient for an immediate implementation, it contains the most important feature of the FK in a very explicit form, namely self-similarity, scalings, and renormalization group equation.

In the following we use the presentation from Montroll and Shlesinger (1984). Consider a random walk on a one-dimensional periodic "lattice" with a spacing one and a probability p_j to make a step of the length a_j. Then probability density to make a step of the length ℓ is

$$P(\ell) = \frac{1}{2}\sum_{j=1}^{\infty} p_j[\delta(\ell - a_j) + \delta(\ell + a_j)] \tag{201}$$

if the random walk is symmetric, and

$$\int_{-\infty}^{\infty} \mathrm{d}\ell\, P(\ell) = 1\ . \tag{202}$$

The crucial idea of the WRW is to consider only scaling type steps and corresponding probabilities, i.e.

$$a_j = a^j, \quad p_j = Cp^j \tag{203}$$

with the normalization constant C

$$C = 1 - p \ . \tag{204}$$

Now $P(\ell)$ appears in the form

$$P(\ell) = \frac{1}{2}(1-p)\sum_{j=0}^{\infty} p^j[\delta(\ell - a^j) + \delta(\ell + a^j)] \ . \tag{205}$$

Using (205), we have for the second moment

$$\langle \ell^2 \rangle = \int_{-\infty}^{\infty} \mathrm{d}\natural^2 P(\ell) = (1-p)\sum_{j=0}^{\infty} (a^2 p)^j \tag{206}$$

which diverges if $a^2 p \geqslant 1$. The characteristic function of $P(\ell)$ is

$$P(k) = \int_{-\infty}^{\infty} \mathrm{d}\ell \ \mathrm{e}^{\mathrm{i}k\ell} P(\ell) = (1-p)\sum_{j=0}^{\infty} p^j \cos(ka^j) \tag{207}$$

which is the Weierstrass function and explains the origin of the name WRW.

The function $P(k)$ satisfies the evident equation

$$P(k) = pP(ka) + (1-p)\cos k \tag{208}$$

The solution of (208) can be presented as a sum

$$P(k) = P_s(k) + P_r(k) \tag{209}$$

of regular (holomorphic) $P_r(k)$ and singular $P_s(k)$ parts. The singular part $P_s(k)$ satisfies the renormalization equation

$$P_s(k) = pP_s(ak) \tag{210}$$

and its solution has a singular behavior at $k = 0$.

Eq. (208) is similar to the *renormalization group equation* (RGE) of the phase transition theory

$$F(g) = \ell^{-d} F(g') + G(g) \tag{211}$$

for the free energy F, interaction constant g, renormalized interaction constant $g' = g'(g)$, regular part G of the free energy, renormalization length ℓ, and the system's dimension d. Nevertheless, Eq. (208) is simpler since its original explicit form (207) permits one to find $P(k)$ explicitly (Shlesinger and Hughes, 1981).

A qualitative analysis of the expression for $P(k)$ is based on the assumption that the form of the singular part $P_s(k)$ has the following behavior near $k = 0$:

$$P_s(k) = |k|^{\mu} Q(k) \tag{212}$$

with some exponent μ and non-singular function $Q(k)$. That is enough to conclude that $0 < \mu < 2$ since the regular part of (207) has only even powers of k, and that $Q(k)$ should be periodic in $\ln k$ with a period $\ln a$ that follows from (210). After substitution of (212) in (210) we also obtain

$$\mu = |\ln p|/\ln a \ . \tag{213}$$

A straightforward way is to apply the inverse Mellins transform to $\cos(a^n k)$ in (207):

$$P(k) = \frac{1-p}{2\pi \mathrm{i}} \sum_{j=0}^{\infty} p^j \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s \, \frac{\Gamma(s)\cos(\pi s/2)}{a^{sj}|k|^s} = \frac{1-p}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s \, \frac{\Gamma(s)|k|^{-s}\cos(\pi s/2)}{1-p/a^s} \tag{214}$$

(see Hughes et al., 1981, 1982; Montroll and Shlesinger, 1984). The numerator has poles at $s = 0, -2, \ldots$, and the denominator has poles at

$$s_j = \ln p/\ln a \pm 2\pi \mathrm{i} j/\ln a = -\mu \pm 2\pi \mathrm{i} j/\ln a \ . \tag{215}$$

The final result from (214) and (215) is obtained by application of the residues over all poles:

$$P(k) = 1 + |k|^{\mu} Q(k) + p \sum_{n=1}^{\infty} \frac{(-1)^n k^{2n}}{(2n)!(1-pa^{2n})} \ ,$$

$$Q(k) = \frac{p}{\ln a} \sum_{n=-\infty}^{\infty} \Gamma(s_n)\cos(\pi s_n/2)\exp(-2\pi \mathrm{i} n \ln|k|/\ln a) \tag{216}$$

in full correspondence with (201) and (213).

We deliberately stop on the details of the WRW since it will be shown in the following sections how many similar features can be found in FK of systems with dynamical chaos. Particularly we have to stress attention on the so-called *log-periodicity* of the singular part $P_s(k)$ through the function $Q(k)$ with respect to the variable k which was well known in the RGT of phase transitions (Niemeijer and van Leeuwen, 1976) but did not play as essential role as it did in the dynamical systems theory (Benkadda et al., 1999; Zaslavsky, 2000a). For other applications of the log-periodicity see Sornette (1998).

7.2. Subdiffusive equations

While the diffusion equation has a fixed form, the form of fractional kinetics is not prescribed and there exists a possibility of numerous fractional formats depending on physical situations. For example, numerous variants of kinetics can appear if we modify expansions (118) and (119) or make special selections of the basic spectral functions $\psi(u), W(q)$ in (183) for the CTRW (Montroll and Shlesinger, 1984; Afanasiev et al., 1991; Chukbar, 1995; Compte, 1996; Saichev and Zaslavsky, 1997; Uchaikin, 1999, 2000).

As an example, instead of (185) consider

$$1/\psi(u) = 1 + u^{\beta}, \quad u \to 0 \quad (0 < \beta \leqslant 1) \ ,$$

$$W(q) = 1 + x_0 q - \frac{1}{2}\sigma q^2 \ . \tag{217}$$

Then $P(q,u)$ satisfies the equation

$$\left(u^\beta + x_0 q + \frac{1}{2}\sigma q^2\right) P(q,u) = u^\beta \tag{218}$$

or in (x,t) variables

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} + x_0 \frac{\partial P(x,t)}{\partial x} = \frac{1}{2}\sigma^2 \frac{\partial^2 P(x,t)}{\partial x^2} + \frac{t^\beta}{\Gamma(1+\beta)}\delta(x) \quad (0 < \beta \leqslant 1) \ . \tag{219}$$

This equation, that corresponds to the fractal Brownian motion (Mandelbrot and Van Ness, 1968), can be generalized to the three-dimensional random walk:

$$\frac{\partial^\beta P(\mathbf{r},t)}{\partial t^\beta} + \mathbf{v}\cdot\nabla P(\mathbf{r},t) = \mathscr{D}\Delta P(\mathbf{r},t) + \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta(\mathbf{r}) \quad (0 < \beta \leqslant 1) \tag{220}$$

(Compte, 1996; Uchaikin, 2000; Weitzner and Zaslavsky, 2001).

As it was mentioned in Section 5.4, for case (219) all moments $\langle x^m \rangle$ are finite. The function $\psi(u)$ in (217) is a characteristic of delays or traps with a probability density to stay time t before the escape

$$\psi(t) \sim 1/t^{1+\beta}, \quad t \to \infty \ . \tag{221}$$

This probability has infinite mean time of escape

$$t_{\mathrm{esc}} \equiv \langle t \rangle = \int_0^\infty t\psi(t)\,\mathrm{d}t \tag{222}$$

for $\beta < 1$, imposing infinite recurrence time. Such a situation should not appear in Hamiltonian dynamics. At the same time, Eq. (219) provides

$$\langle x^2 \rangle = \frac{\sigma}{\Gamma(1+\beta)} t^\beta \quad (0 < \beta \leqslant 1) \tag{223}$$

that corresponds to the subdiffusion (Saichev and Zaslavsky, 1997). The Hamiltonian phase space is at least two-dimensional, and the subdiffusion along one variable and superdiffusion or diffusion along the other one are not forbidden. Just this case of subdiffusion in Hamiltonian dynamics was observed in Afanasiev et al. (1991), Schwägerl and Krug (1991) and Benenti et al. (2001).

7.3. Superdiffusion

In FK the second moments $\langle \xi^2 \rangle$ of some physical variable ξ can be infinite as it is for the Lévy processes. In a fairly general situation we can use expression (140)

$$\langle |\xi| \rangle \sim t^{\beta/\alpha} = t^{\mu/2} \ , \tag{224}$$

where we replace x by an abstract ξ. The transport exponent μ (see (141)) is defined in a way as the second moment is finite,

$$\langle \xi^2 \rangle \sim t^\mu \ . \tag{225}$$

The case with $\mu = 1$ corresponds to the normal diffusion, $\mu < 1$ is subdiffusion, and $\mu > 1$ is superdiffusion. All these definitions assume self-similarity if we use definition (224), and a possibility

of using (225) at least for the truncated moments, i.e.

$$\langle \xi^2 \rangle_{\rm tr} \sim t^{\mu} \tag{226}$$

(compare to (169)). Dynamical systems, even fairly simple at first glance, display a rich set of different possibilities: the value of μ can be very close to one, but the dynamics can be too far from the Gaussian process, self-similarity can be valid in some restrictive way and can be applied not for all moments, etc. We will meet these examples in the following sections.

A simplified version of superdiffusion can be introduced directly as (134) or (139)

$$\frac{\partial P(\xi,t)}{\partial t} = \mathscr{A}\,\frac{\partial^{\alpha} P(\xi,t)}{\partial |\xi|^{\alpha}} \tag{227}$$

or

$$\langle |\xi|^{\alpha} \rangle = \Gamma(1+\alpha)\mathscr{A} t \ , \tag{228}$$

where we put $\mathscr{A} = \text{const.}, \beta = 1$, and replace x by ξ. Expression (228) implies in the case of self-similarity

$$\mu = 2/\alpha > 1 \tag{229}$$

since $\alpha < 2$. A similar result comes from (153) if we put $\beta = 1$.

All these situations were considered in numerous publications (Montroll and Shlesinger, 1984; Afanasiev et al., 1991; Fogedby, 1994; Chukbar, 1995; Compte, 1996; Saichev and Zaslavsky, 1997). We should comment that the superdiffusion can also exist for $\beta < 1$. An interesting situation appears in the case when there are few different directions of diffusion. In Hamiltonian dynamics there are at least two directions: along the coordinate and along the momentum. In such a case

$$\langle |\xi_{\mathbf{e}^{(0)}}| \rangle \sim \mathscr{D}_{\mathbf{e}^{(0)}} t^{\mu(\mathbf{e}^{(0)})/2} \ , \tag{230}$$

i.e. the transport exponent μ depends on the direction of transport, given by the unit vector $\mathbf{e}^{(0)}$ (Chernikov et al., 1990; Petrovichev et al., 1990; Weitzner and Zaslavsky, 2001; Meerschaert and Scheffler, 2000; Uchaikin, 1998). In this way the FK imposes a new type of the anisotropy which reveals not only in the diffusion constant $\mathscr{D}_{\mathrm{e}^{(0)}}$ but also in the transport exponent $\mu(\mathrm{e}^{(0)})$. It follows from the Kac lemma that at least in one direction should be the superdiffusion (see also a comment at the end of the previous section.

Generalization of the FK for the anisotropic case can be performed in different ways (Zolotarev et al., 1999; Weitzner and Zaslavsky, 2001).

The simplest form (Compte, 1996; Yanovsky et al., 2000; Zolotarev et al., 1999) is to use the Riesz fractional derivative for the fractional Laplace operator

$$(-\Delta)^{\alpha/2} \xrightarrow{F} |\mathbf{k}|^{\alpha} \tag{231}$$

defined through its Fourier transform. Explicit expression for the fractional Laplacian is more complicated (see the Appendix B). Using expression (231) one can write down the generalized FKE

$$\frac{\partial^{\beta} F(\mathbf{r},t)}{\partial t^{\beta}} = \mathscr{D}_{\alpha\beta}(-\Delta)^{\alpha/2} f(\mathbf{r},t) \ . \tag{232}$$

Different asymptotic solutions for (232) are given in Zolotarev et al. (1999) and Weitzner and Zaslavsky (2001).

8. Renormalization group of kinetics (GRK)

Renormalization group (RG) is a powerful tool to study physical phenomena with scaling properties (see for example in Kadanoff, 1981, 1993, 2000; Greene et al., 1981; MacKay, 1983; Hentschel and Procaccia, 1983; Jensen et al., 1985). Initially, the methods of RG proved to be efficient in the field theory and statistical physics and later they were applied to dynamics and chaos. The Gaussian process can be considered as a first and trivial example of a possible application of RG theory in the kinetics, and crucial achievements were obtained for the kinetics near a phase transition point (Wilson and Kogut, 1974; Hohenberg and Halperin, 1977). Chaotic dynamics near the so-called cantori exhibits scaling properties and the corresponding kinetic equation was derived by Hanson et al. (1985) (see also MacKay et al., 1984). An important feature of RGK is the coupling of space–time scalings. This coupling has pure dynamical origin and it imposes a necessity to go deeply into understanding of the dynamical singularities (see Sections 2.4 and 2.5) in order to find the links between spatial and temporal behavior of trajectories. The material of this section follows Zaslavsky (1984a, b), Zaslavsky (2000a) and Benkadda et al. (1999).

8.1. Space–time scalings

Let ℓ be a space variable and t be the time variable so that $P(\ell,t)$ is a density distribution along the dynamical trajectories. Consider an infinitesimal change of $P(\ell,t)$ in time, i.e.

$$\delta_t P(\ell,t) = P(\ell,t+\Delta t) - P(\ell,t) \tag{233}$$

and the corresponding to Δt infinitesimal change in space

$$\delta_\ell P(\ell,t) = \sum_{\Delta\ell} [P(\ell+\Delta\ell,t) - P(\ell,t)] \ , \tag{234}$$

where the sum is performed over all possible paths of the displacement $\Delta\ell$ during the same time interval Δt. Fractal properties of the chaotic trajectories can be applied directly to (233) and (234)

$$\delta_t P = (\Delta t)^\beta \frac{\partial^\beta P}{\partial t^\beta} \ ,$$

$$\delta_\ell P = \sum_{\Delta\ell} \frac{\partial^\alpha}{\partial \ell^\alpha} (|\Delta\ell|^\alpha \mathscr{A}' P) \ , \tag{235}$$

where α, β are critical exponents that characterize the fractal structures of space–time, ℓ is assumed to be positive (length of a "flight"), and $\mathscr{A}'$ is a function of ℓ and other phase space variables additional to ℓ.

A typical kinetic equation appears as a balance equation

$$\delta_t P = \overline{\delta_\ell P} \ , \tag{236}$$

where the bar means averaging over all possible paths and additional to ℓ variables. Using a notation

$$\mathscr{A} \equiv \sum_{\Delta\ell} \overline{\frac{|\Delta\ell|^\alpha}{(\Delta t)^\beta} \mathscr{A}'} \tag{237}$$

we arrive from (236) to

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^\alpha}{\partial \ell^\alpha}(\mathscr{A}P) \tag{238}$$

i.e. to a simplified version of the FKE (134) with $\ell = |x|$.

The RGK can be introduced starting from the dynamics. Let the dynamics carry the rescaling properties

$$\hat{R}_K : \Delta\ell \to \lambda_\ell \Delta\ell, \quad \Delta t \to \lambda_T \Delta t \ , \tag{239}$$

which one may accept in a broad sense, i.e. the rescaling exists after some averaging, coarse-graining, in restricted time–space domains, etc. The basic feature of the RGK is that the balance equation (236) is invariant under the renormalization transform (239), i.e.

$$\hat{R}_K \delta_t P = \hat{R}_K \overline{\delta_\ell P} \tag{240}$$

which means that

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\lambda_\ell^\alpha}{\lambda_T^\beta} \frac{\partial^\alpha}{\partial \ell^\alpha}(\mathscr{A}P) \tag{241}$$

Starting from some minimal values Δt, $\Delta\ell$, one can apply $\hat{R}_K$ arbitrary number n times. This transforms (241) into

$$\frac{\partial^\beta P}{\partial t^\beta} = \left(\frac{\lambda_\ell^\alpha}{\lambda_T^\beta}\right)^n \frac{\partial^\alpha}{\partial \ell^\alpha}(\mathscr{A}P) \ . \tag{242}$$

The kinetic equation (238) "survives" under action of $\hat{R}_K$ if the limit $n \to \infty$ has a sense, i.e. if there exists a limit

$$\lim_{n\to\infty}(\lambda_\ell^\alpha/\lambda_T^\beta)^n = 1 \tag{243}$$

which leads to the fixed point solution

$$\beta/\alpha = \ln \lambda_\ell / \ln \lambda_T \equiv \mu/2 \ . \tag{244}$$

Eq. (244) shows an important connection between the exponents (α, β) and scaling constants $(\lambda_\ell, \lambda_T)$. This result can be expanded to a more fundamental property if we consider the moment equation for (238), i.e.

$$\langle \ell^\alpha \rangle = \mathscr{D}_{\alpha\beta} t^\beta \tag{245}$$

(compare to (139)) with

$$\mathscr{D}_{\alpha\beta} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)} \mathscr{A} \ . \tag{246}$$

Comparing (244) to (141) we obtain that μ is the transport exponent

$$\mu = 2 \ln \lambda_\ell / \ln \lambda_T \tag{247}$$

which is now expressed through the scaling parameters $(\lambda_\ell, \lambda_T)$.

8.2. Log-periodicity

The existence of FKE is based on the formal existence of the limit

$$\lim_{\Delta t\to 0} \overline{\mathscr{A}'|\Delta\ell|^{\alpha}}/(\Delta t)^{\beta} = \text{const.} \tag{248}$$

which is equivalent to the Kolmogorov condition for the FPK equation

$$\lim_{\Delta t\to 0} \overline{(\Delta\ell)^2}/\Delta t = \text{const.} \tag{249}$$

The essential difference, in addition to the difference of the exponents, appears when we impose invariance of the FKE under the RG-transform $\hat{R}_K$. Indeed, expression (249) is invariant under the transform

$$\hat{R}_G : \Delta\ell \to \lambda^{1/2}\Delta\ell, \quad \Delta t \to \lambda\Delta t \tag{250}$$

with arbitrary value of λ. That means that the Gaussian renormalization $\hat{R}_G$ is continuous, while the $\hat{R}_K$ is discrete with fixed values of λ_ℓ and λ_T. As a result, a new property of the solution of FKE occurs known as log-periodicity (see a review article on log-periodicity by Sornette (1998) and application to FKE in Benkadda et al. (1999) and Zaslavsky (2000a)).

The FKE (238) is linear and a sum of its solutions is also a solution. Particularly, fixed-point equation (243) has infinite number of solutions. For example,

$$\beta_k = \alpha \ln \lambda_\ell / \ln \lambda_T + 2\pi i k / \ln \lambda_T = \frac{1}{2}\alpha\mu + 2\pi i k/\ln\lambda_T \quad (k = 0, \pm 1, \ldots) \tag{251}$$

satisfies (243) and expression (244) gives a particular solution only for $k = 0$. Using (251), we can rewrite the moment equation (245) in the form of the superposition

$$\langle \ell^{\alpha} \rangle = \sum_{k=0}^{\infty} \mathscr{D}_{\alpha\beta}^{(k)} t^{\beta_k} = \mathscr{D}_{\alpha\beta}^{(0)} t^{\beta} \left\{ 1 + 2\sum_{k=1}^{\infty} (\mathscr{D}_{\alpha\beta}^{(k)}/\mathscr{D}_{\alpha\beta}^{(0)}) \cdot \cos\left(\frac{2\pi k}{\ln\lambda_T}\ln t\right)\right\}, \tag{252}$$

where new coefficients $\mathscr{D}_{\alpha\beta}^{(k)}$ $(k > 0)$ are unknown and typically are small. The term in braces is periodic with respect to $\ln t$ and the period is

$$T_{\log} = \ln \lambda_T \ . \tag{253}$$

Fig. 9 provides an example of log-periodicity for the web map of the 4th order symmetry. More examples appear in the forthcoming section.

For the cases when the initial kinetic equation has a discrete renormalization invariance in an explicit form the coefficients $\mathscr{D}_{\alpha\beta}^{(k)}$ can be explicitly calculated (Hanson et al., 1985; Montroll and Shlesinger, 1984; see also Section 7.1 on WRW). Another comment is related to the limit $\Delta t \to 0$ in (248). When the RG is discrete, there are cut-off values of $\min \Delta\ell \equiv \Delta\ell_0$ and $\min \Delta t \equiv \Delta t_0$, and all other permitted values of $\Delta\ell, \Delta t$ are greater or much greater of $\Delta\ell_0, \Delta t_0$. A physical meaning of them will be given in Section 9. Some other application of the log-periodicity can be found in Shlesinger and West (1991).

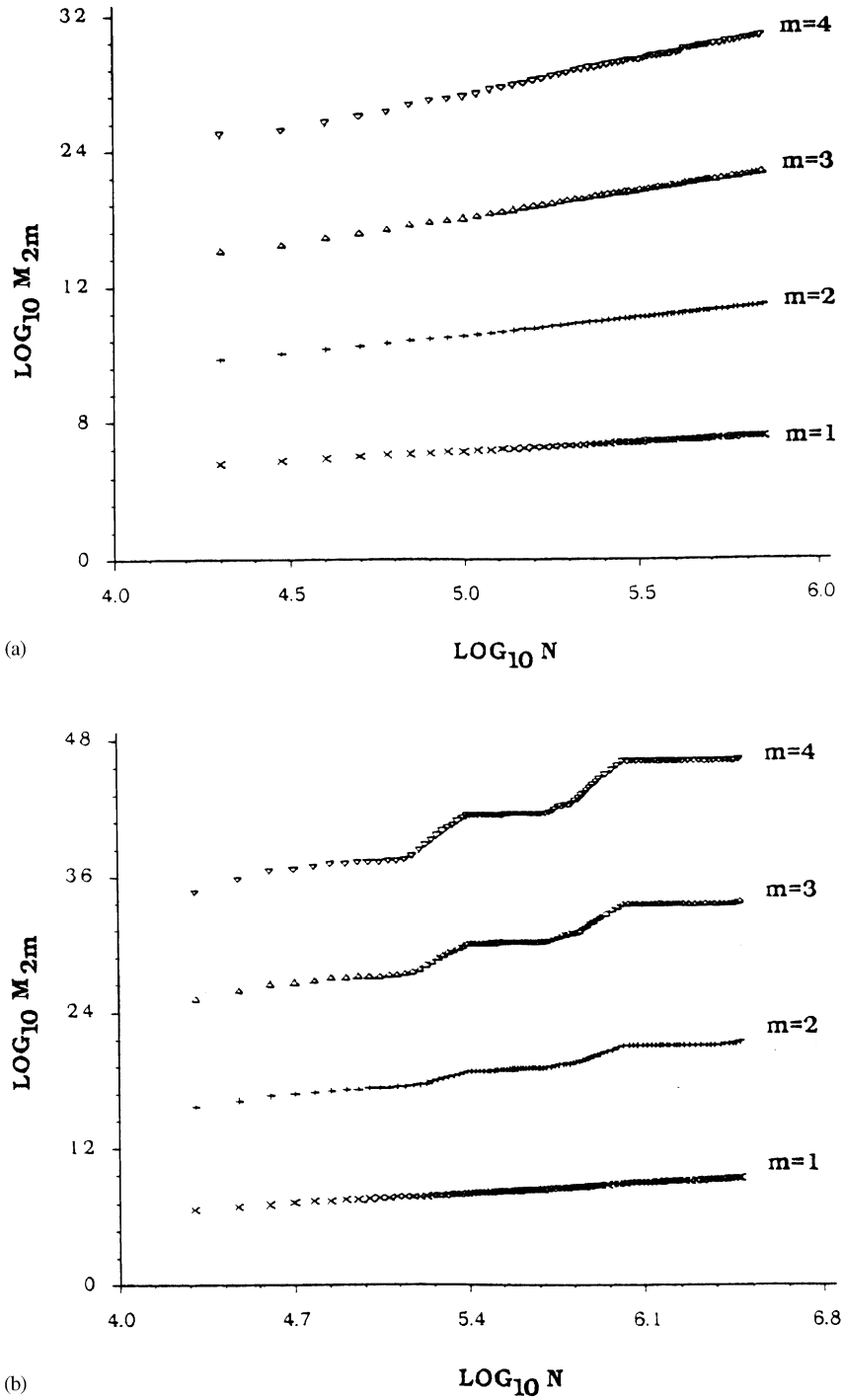

Fig. 9. Log-log plots of the even moments $M_{2m} = \langle R^{2m} \rangle$ of the trajectories displacement R vs. number of iterations (time) N for the web map with 4-fold symmetry and $K = 6.25$, top, with normal diffusion, and $K = 6.349972$, bottom, with superdiffusion (Zaslavsky and Niyazov, 1997).

8.3. Multi-scaling

For a general dynamics of even low-dimensional systems, calculations with only two scaling exponents α and β is a fairly strong simplification. Multi-scaling dynamics is a more adequate form of dynamics, and the log-periodicity can be considered as an example of the reduced multi-fractality. Multi-scaling can occur either from a singular zone or from several singular zones. A general idea of the multi-fractality was developed in Grassberger and Procaccia (1984), Parisi and Frisch (1985), Jensen et al. (1985), Halsey et al. (1986) and Paladin and Vulpiani (1987).

It is assumed the existence of many different exponents, say different β, with a distribution function $\rho(\beta)$. We can use again the property of linearity of the FKE and consider its solution as $P(\ell, t; \beta)$ and the normalization condition

$$\int_{\ell_{\min}=\Delta\ell_0}^{\infty} \mathrm{d}\ell \int \mathrm{d}\beta\, \rho(\beta) P(\ell, t; \beta) = 1 \ . \tag{254}$$

Expression (245) for the moments should be modified as

$$\langle \ell^{\alpha} \rangle = \int_{\Delta\ell_0}^{\infty} \mathrm{d}\ell \int \mathrm{d}\beta \ \rho(\beta) \mathscr{D}_{\alpha\beta} t^{\beta} P(\ell, t; \beta) \ . \tag{255}$$

All the following steps are fairly formal depending on the functions $\rho(\beta)$, $\mathscr{D}_{\alpha\beta}^{(k)}$, etc., and as in many typical situations, properties of $\rho(\beta)$ permit to apply the steepest descent method to perform and integration over β. This procedure will select a special value β (or a few of them) for which the under-integral extremum appears (see for details in Zaslavsky, 2000b).

9. Dynamical traps

9.1. General comments and definitions

In Sections 2.4 and 2.5 we have preliminary discussions of such properties of dynamics as singular zones and stickiness. This section considers these notions in more details using a more comprehensive phenomenology of *dynamical traps*. We pursue an idea that kinetics and transport of chaotic systems, in general, can be evaluated when the phase space topology is fairly well known. In other words, while a "microscopic" theory of kinetics in statistical physics is based on our knowledge about the acting forces, in the kinetic theory of dynamical chaos, one should go further and deeper and understand the phase space topology that depends on the system's parameters values and that can vary for the same type of the acting forces. Dynamical trap is, perhaps, the most important part of the topology-transport connection, that is a long standing problem of physics and mathematics (Hanson et al., 1985; Meiss and Ott, 1985; Meiss, 1992; Wiggins, 1992; Rom-Kedar et al., 1990; Haller, 2000, 2001a, b; Melnikov, 1996; Zaslavsky, 1984; Chernikov et al., 1990; Rom-Kedar and Zaslavsky, 1999; Zaslavsky, 2002).

The trap, in its absolute meaning, does not exist in Hamiltonian dynamics due to the Liouville theorem on the phase volume preservation. In Section 2.3 we have introduced the distribution function $P(A; t)$ of Poincaré recurrences to a domain A. Dynamical trap of an order $m_0 > 1$ is defined as

a domain A_{m_0} such that

$$\tau_m(A_{m_0}) \equiv \langle t^m \rangle = \frac{1}{\Gamma(A_{m_0})} \int_0^\infty t^m P(A_{m_0}, t)\, \mathrm{d}t = \infty \quad (m \geqslant m_0 > 1) \ , \tag{256}$$

where the value m_0 may be a non-integer. The domain A_{m_0} is a singular zone. A system may have many singular zones and one can expect the asymptotics of the (21) type

$$P_{\mathrm{rec}}(A_{m_0}, t) \sim \frac{1}{t^\gamma} \tag{257}$$

with $\gamma = \gamma(A_{m_0})$. For the normalized distribution of recurrences $P_{\mathrm{rec}}(t)$ (see (14)), one can write

$$P_{\mathrm{rec}}(t) \sim \sum_j c_j / t^{\gamma(A_j)} \tag{258}$$

indicating that different singular zones with their own recurrence exponents make input into the final asymptotics. As $t \to \infty$, the lowest value of the recurrence exponent γ for $P_{\mathrm{rec}}(t)$ survives while for different time intervals can be different intermediate asymptotics and different values of γ (see Sections 2.3 and 2.4). It can also be more complicated non-algebraic forms of $P_{\mathrm{rec}}(t)$.

There is no classification of singular zones or dynamical traps. A general comment is that any boundaries or singular sets (singular points) can play such a role. In this section we provide a qualitative description of three types of possible traps (Zaslavsky, 2002) and estimate their relation to the kinetics and transport problems.

9.2. Tangle islands

The islands of the so-called accelerator mode and ballistic mode were called tangle islands in Rom-Kedar and Zaslavsky (1999) to distinguish them from the resonance islands. Stickiness of trajectories to the boundaries of tangle islands leads to the superdiffusion (Zaslavsky and Niyazov, 1997; Zaslavsky et al., 1997; Iomin et al., 1998). Here we provide a few examples of the tangled islands (see also Sections 2.4 and 2.5).

9.2.1. Accelerator mode island (standard map)

For the standard map (8) an accelerator mode island appears for K in the intervals

$$2\pi m \leqslant K \leqslant 2\pi m + \Delta K(m) \ , \tag{259}$$

where m is an integer and $\Delta K(m)$ is an interval of the existence of the corresponding island. There are other intervals of K where there are accelerator mode islands of higher orders (see more in Lichtenberg and Lieberman, 1983). For the values of K in (259), starting from $x_0 = \pi/2$, $p_0 = 0$, a particle moves with the linearly growing value of p_n:

$$p_n = 2\pi m \pi \tag{260}$$

that explains the name of the island. The island collapses at

$$K = [4^2 + (2\pi m)^2]^{1/2} \ . \tag{261}$$

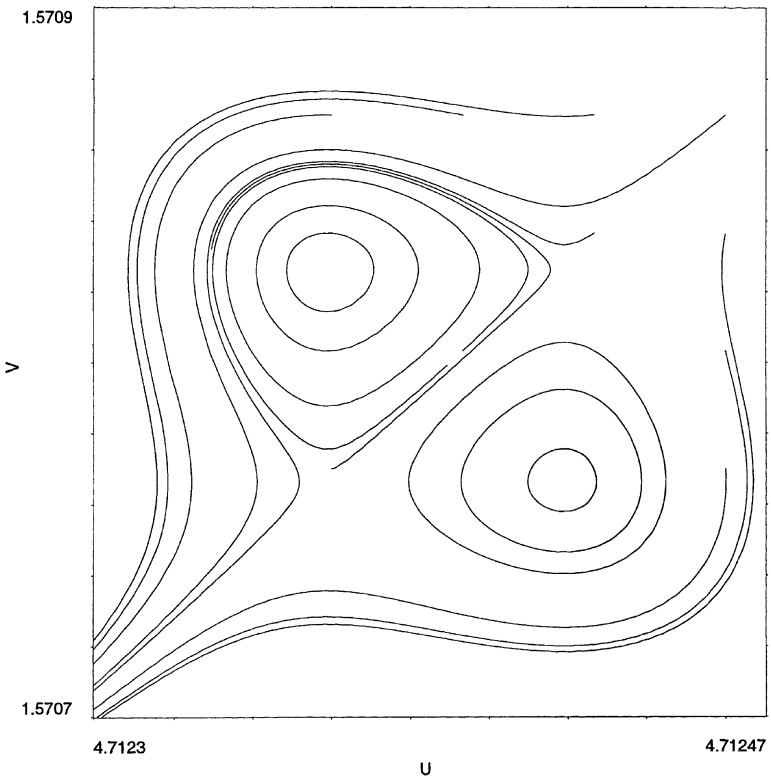

Fig. 10. Accelerator mode islands for the web map with $K = 6.28318531$.

An effective Hamiltonian of regular trajectories inside the island can be written as (Karney, 1983)

$$H^{(a)} = \frac{1}{2}(\Delta p)^2 + (K - K_c)\Delta x - \frac{\pi}{3}(\Delta x)^3$$

$$K_c = 2\pi m, \quad \Delta x = x - x^{(a)}, \quad \Delta p = p - p^{(a)} \tag{262}$$

and $x^{(a)}$, $p^{(a)}$ represents the trajectory with $K = K_c, x_0 = \pi/2$, $p_0 = 0$.

9.2.2. Accelerator mode island (web map)

It is a similar type of islands for map (11). An example of the value of K_c is

$$K_c = 2\pi, \quad u_0^{(a)} = \pi/2, \quad v_0^{(a)} = 3\pi/2 \tag{263}$$

which generates the trajectory $(u_n^{(a)}, v_n^{(a)})$. The effective Hamiltonian is (Zaslavsky and Niyazov, 1997)

$$H^{(a)} = \frac{1}{2}(K - K_c)(\Delta v - \Delta u) - \frac{\pi}{6}[(\Delta v)^3 - (\Delta u)^3] \tag{264}$$

(see Fig. 10). The islands appear for $K > K_c$ and correspond to some degenerate case.

9.2.3. Ballistic mode island (separatrix map)

This example appears in Rom-Kedar and Zaslavsky (1999) for the separatrix map (Zaslavsky and Filonenko, 1968):

$$h_{n+1} = h_n + \epsilon K_n \sin\phi_n \ ,$$

$$\phi_{n+1} = \phi_n + v \ln(32/|h_{n+1}|), \quad (\mathrm{mod}\ 2\pi) \tag{265}$$

with

$$K_n = \begin{cases} K = 8\pi v^2 \exp(-\pi v/2), & \sigma_n = 1 \ , \\ 0, & \sigma_n = -1 \ , \end{cases}$$

$$\sigma_{n+1} = \sigma_n \ \mathrm{sign}\ h_{n+1} \tag{266}$$

for a model equation of the perturbed pendulum

$$\ddot{x} + \sin x = \epsilon \sin(x - vt) \tag{267}$$

(for the equations in form (265) and (266) see in Rom-Kedar (1994)). The basic ballistic trajectory is

$$\sigma_0 = 1, \quad h_0^* = \epsilon^* K^*/2, \quad \phi_0 = 3\pi/2 \tag{268}$$

with an additional condition

$$\epsilon^* K^* = 64 \exp\left(-\frac{\pi}{2v^*}(2m+1)\right), \quad m = 0, 1, \ldots \ . \tag{269}$$

Then

$$h_4^* = h_0^*, \quad \phi_4^* = \phi_0^* + (4m+2)\pi = \phi_0 \quad (\mathrm{mod}\ 2\pi) \ ,$$

$$\sigma_4^* = \sigma_0$$

and the dynamics is ballistic with the period (see also in Iomin et al. (1998)). The effective Hamiltonian for trajectories within the ballistic island, that deviates from (268) and (269), is

$$H^{(\mathrm{bal})} = h^*(\Delta\psi)^2 + 4v\frac{\epsilon - \epsilon^*}{\epsilon^*}\Delta h - \frac{2}{3}\frac{v^3}{h^{*2}}(\Delta h)^3 \tag{270}$$

with

$$\Delta\psi_k = \Delta\phi_k + \frac{v}{h^*}\Delta h_k, \quad \Delta h_k = h_k - h_k^*, \quad \Delta\phi_k = \phi_k - \phi_k^* \ . \tag{271}$$

The island exists for $\epsilon > \epsilon^*$.

The structure of Hamiltonians (262), (264) and (270) defines behavior of trajectories near the island boundaries and, in this way, intermittent part of chaotic trajectories.

9.3. Hierarchical islands trap (HIT)

This type of a trap was introduced and described in Zaslavsky (1995) and Zaslavsky and Edelman (1997), although its demonstrations for the standard map and web map appeared later (Zaslavsky et al., 1997; Benkadda et al., 1997a, b; White et al., 1998). Fig. 11 shows hierarchical set of domains

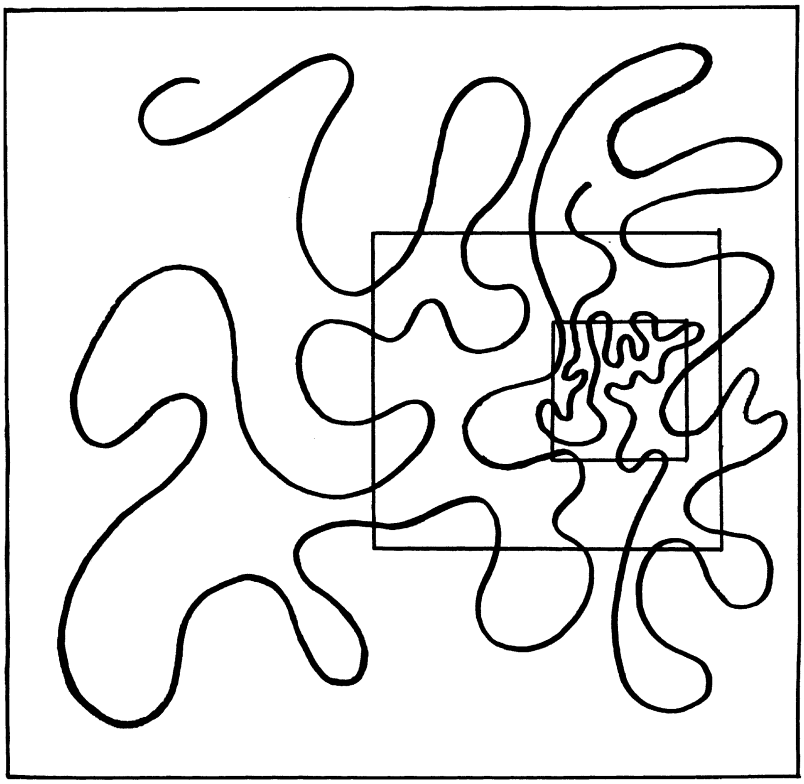

Fig. 11. Model of the dynamical quasi-trap (stretched time).

with "stretched time". The smaller is a sub-domain, the longer trajectories stay there. An example of such phenomena was given in Section 2, Fig. 6 for the standard map. Another example is in Fig. 12 for the web map. A formal description of the HIT is the following.

Consider a set of domains embedded into each other: $A_0 \supset A_1 \supset \cdots \supset A_n \supset \cdots$, and their differences $A_0 - A_1$, $A_1 - A_2, \ldots, A_n - A_{n-1}, \ldots$, such that $\Gamma_n \equiv \Gamma(A_n - A_{n+1})$ is the phase volume of $A_n - A_{n-1}$. Let us introduce a characteristic residence time T_n that a trajectory spends in Γ_n during the time interval between the first arrival to $\Delta A_n \equiv A_n - A_{n+1}$ and the first following departure from ΔA_n. The HIT is defined by the following scaling properties:

$$\Gamma_{n+1} = \lambda_\Gamma \Gamma_n \quad (\lambda_\Gamma < 1) \ ,$$

$$T_{n+1} = \lambda_T T_n \quad (\lambda_T > 1) \ . \tag{272}$$

Let us point out that, contrary to the systems with uniform mixing and with residence time proportional to the phase volume of a considered domain, the conditions $\lambda_T > 1$, $\lambda_\Gamma < 1$ are exactly the opposite ones: the smaller is the domain ΔA_n, the longer the trajectories stay there. Conditions (272) reflect not only the presence of a singular zone, but also its self-similar structure.

An exact self-similarity that follows from (272), i.e.

$$T_n = \lambda_T^n T_0, \quad \Gamma_n = \lambda_\Gamma^n \Gamma_0 \tag{273}$$

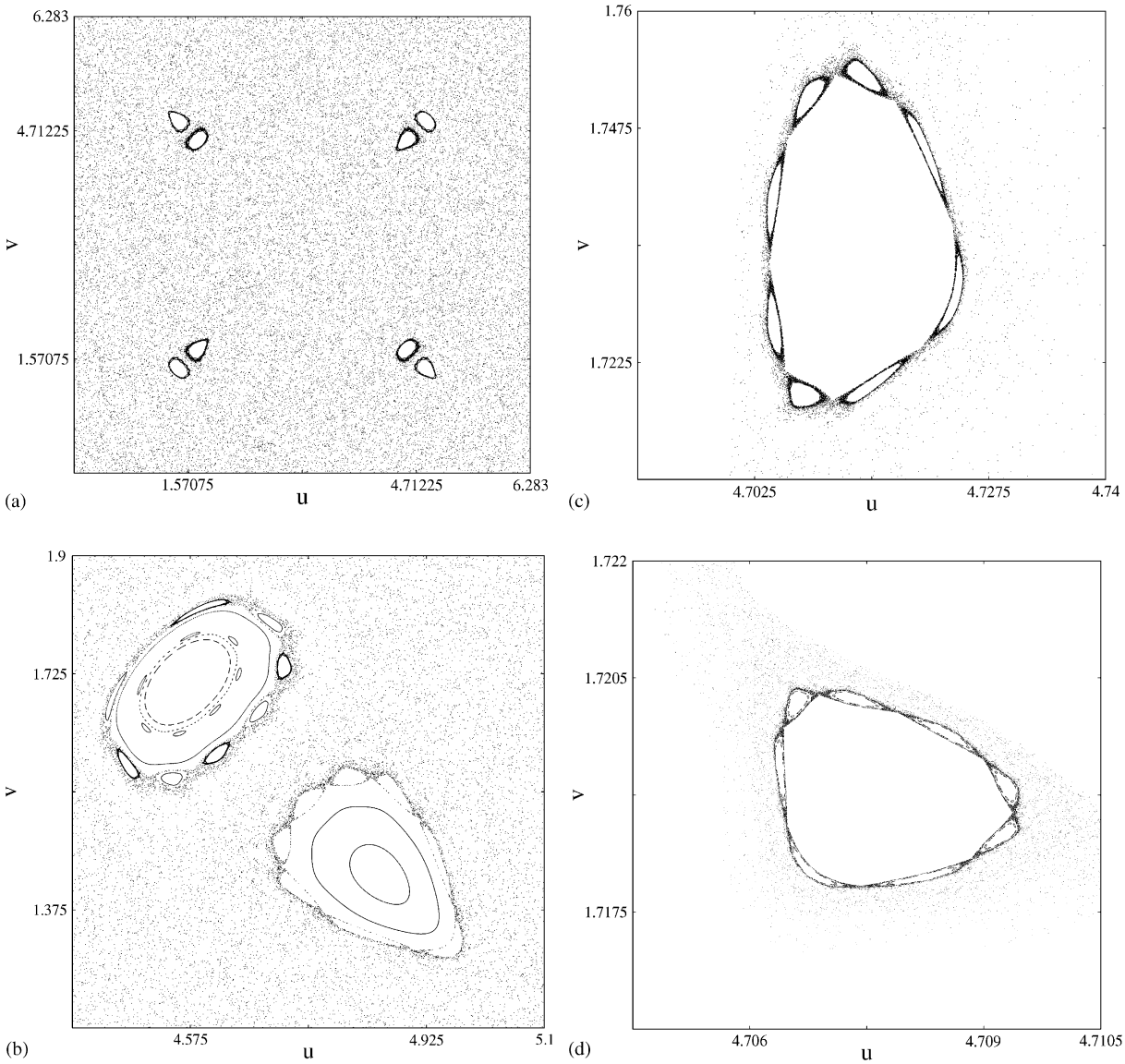

Fig. 12. HIT for the web map ($K = 6.349972$). Figures represent a sequence of the magnifications of the set of islands-around-islands (Zaslavsky and Niyazov, 1997).

is a kind of an idealization, and it can appear only for special values of a control parameter, say K^*, obtained with very high accuracy. More typical is the situation when the scaling parameters in (272) depend on n, i.e. instead of (273) we have

$$\Gamma_n = \bar{\lambda}_\Gamma^n \Gamma_0, \;\; T_n = \bar{\lambda}_T^n T_0 \quad (n \to \infty) \;, \tag{274}$$

where $\bar{\lambda}_\Gamma$ and $\bar{\lambda}_T$ are some effective values defined by the equations

$$\ln \bar{\lambda}_\Gamma = \frac{1}{n}\sum_{j=1}^{n} \ln \lambda_\Gamma(j) < 0 \quad (n \to \infty) \ ,$$

$$\ln \bar{\lambda}_T = \frac{1}{n}\sum_{j=1}^{n} \ln \lambda_T(j) > 0 \quad (n \to \infty) \tag{275}$$

with scaling constants $\lambda_\Gamma(j), \lambda_T(j)$ on jth step.

A similar to (272) scaling can be proposed for the Lyapunov exponent

$$\sigma = \lim_{t\to\infty} \lim_{\delta x(0)\to 0} \frac{1}{t} \ln \left| \frac{\delta x(t)}{\delta x(0)} \right| \ , \tag{276}$$

where $\delta x(t)$ is the distance in phase space between two trajectories at time t. In fact, one should introduce "finite-size" Lyapunov exponent (Aurell et al., 1997) $\sigma(\Delta x, \Delta t)$ which depends on space–time windows $\Delta x, \Delta t$). Then one can write (see Zaslavsky, 1995):

$$\sigma_n = \sigma(\Delta t = T_n) = \lambda_T^n \sigma_0, \quad (x \in \Delta x_n) \ . \tag{277}$$

In Fig. 12, Δx_n is a width of the strip (boundary layer) around an island of the nth generation. For construction (272) Δx_n can be the diameter of $A_n - A_{n+1}$.

9.4. Stochastic net-trap

An example of this trap was shown in Fig. 5 (see more in White et al., 1998; Kassibrakis et al., 1998; Zaslavsky and Edelman, 2002). The stochastic net is a dark grid that appears in Fig. 5 and where a trajectory spends longer time than outside of the domain. The topology in Fig. 5 can appear in the same systems as the HIT for some special values of the control parameter. Theory of this type of topology does not exist. Nevertheless some transport properties for this situation can be conjectured (see below).

9.5. Stochastic layer trap

The stochastic layer trap was discussed in (Petrovichev et al., 1990) for the problem of an particle advection in Beltrami flow, and later in Rakhlin (2000) for the problem of electron dynamics in a magnetic field and periodic lattice. The main features of the stochastic layer trap can be understood from Fig. 13. Let an island with one elliptic point is embedded into the stochastic sea. After a change of the control parameter, a bifurcation appears that creates two additional elliptic and hyperbolic points. The separatrices are destroyed (split) by the perturbation and they are replaced by an exponentially narrow stochastic layer. As we continue to change the control parameter, the width of the stochastic layer becomes larger. An important feature of the pattern in Fig. 13 is that the stochastic layer is separated from the stochastic sea. By a small change of the control parameter, we can achieve a merge of the stochastic layer to the stochastic sea as on the bottom of Fig. 13. A border of the merge can be arbitrarily small and therefore the probability of crossing the border can be arbitrarily small. The domain of the former stochastic layer is a trap, and a trajectory, that

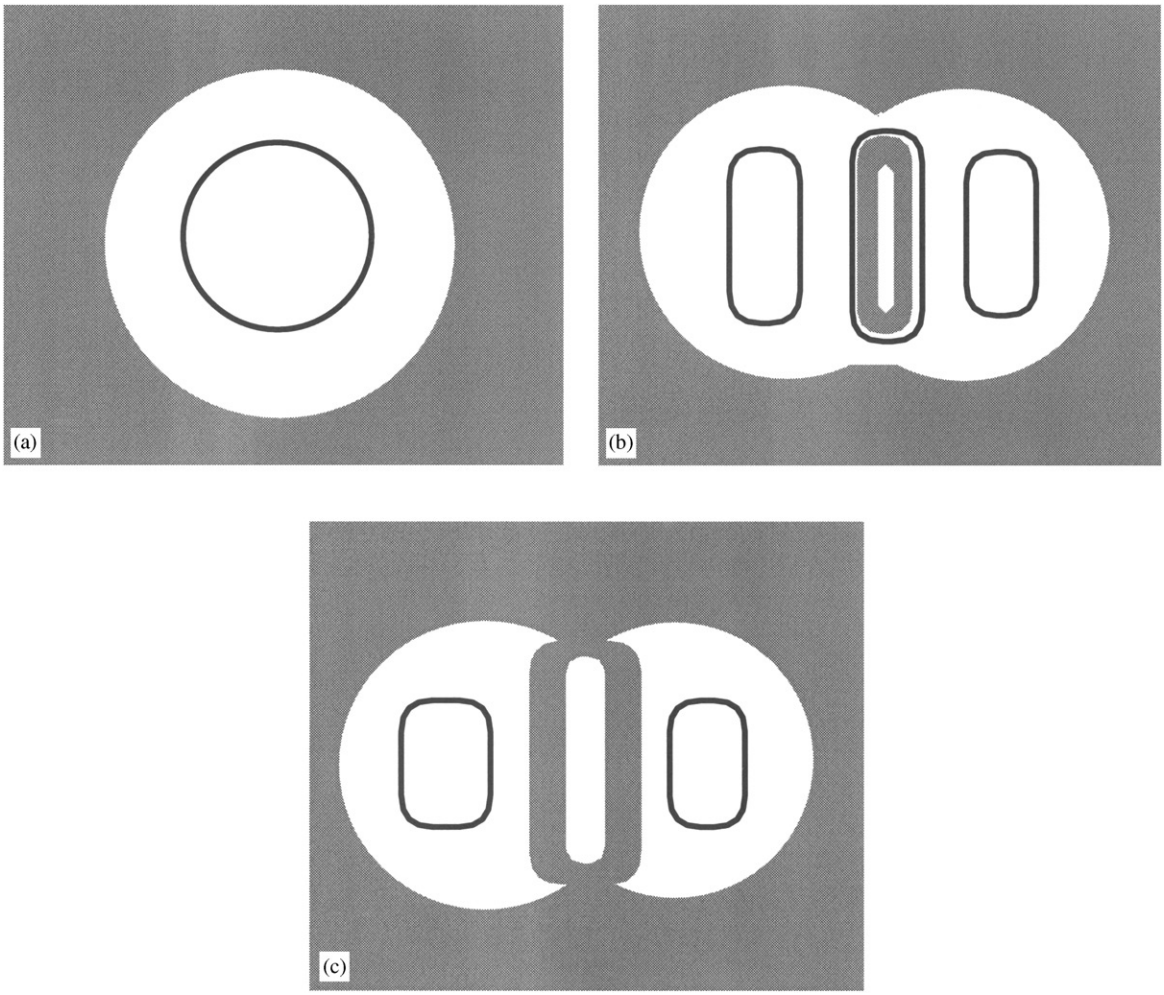

Fig. 13. Appearance of the stochastic layer trap as a result of the doubling bifurcation.

enters the trap can spend there an astronomical time. There is no quantitative description of this type of trap.

Some recent results on the so-called strong anomalous transport (Aurell et al., 1997; Castiglioni et al., 1999) can be understood just as an occurrence of the stochastic layer trap. A simulation for the standard map and $K = K_V = 6.9115$ performed in Aurell et al. (1997) shows for the transport exponent

$$\mu(m) \approx 2m, \quad (m > 3) \tag{278}$$

i.e. the value that corresponds to the pure ballistic dynamics. The phase portrait that corresponds to $K = K_V$ is given in Fig. 14 (Zaslavsky, 2002). While the part (a) of Fig. 14 does not show a big difference from the part (a) of Fig. 6, the sequence of resonances, that is resolved in Fig. 14, is 1–3–7–9–10 and after that we see in the part (d) of Fig. 14a situation similar to the net-trap. The

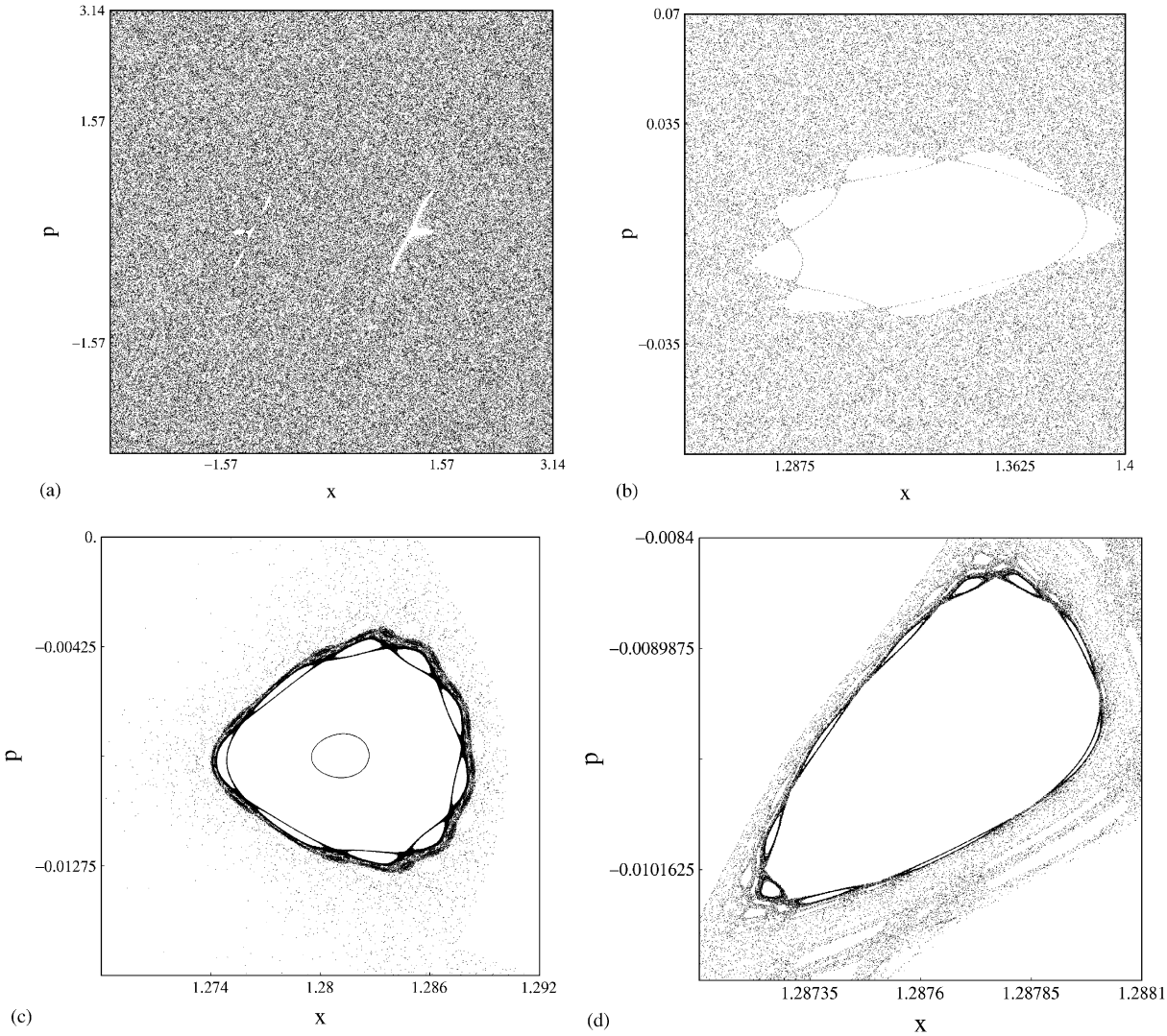

Fig. 14. Stochastic layer trap for the standard map ($K = 6.9115$).

dark strip in (c) and (d) of Fig. 14 corresponds to a trajectory trapped for extremely long time near a high order islands-chain. The simulations show the escape from the strip after 10^6–10^7 iterations. This means that we need at least two orders more trajectories (i.e. $N \sim 10^8$–10^9) to find an escape exponent $\gamma_{\rm esc}(A)$ for the part A of phase space that corresponds to the stochastic layer (dark strip area), in Fig. 14, while it was only about $N \sim 10^6$ in Aurell et al. (1997) (see also Ferrari et al., 2001). Indeed, a simple estimate is

$$\langle p^{2m} \rangle \sim \langle p^{2m} \rangle_{\rm anom} + \langle p^{2m} \rangle_{\rm ball} \ , \tag{279}$$

where the first term corresponds to the trajectories and their parts that participate in the anomalous diffusion, and the second term corresponds to the ballistic trajectories. Rewriting (279) as

$$\langle p^{2m} \rangle \sim N t^{\mu(m)} + \delta N \cdot t^{2m} \tag{280}$$

with δN as a number of ballistic trajectories, we have a condition when the second term suppresses the non-ballistic diffusion:

$$t^{2m-\mu(m)} \gtrsim N/\delta N \sim N \tag{281}$$

or

$$(2-\mu(1))m \gtrsim \ln N/\ln t \ . \tag{282}$$

For $t \sim 10^7$, $N \sim 10^6$, and $\mu(m) \sim 1.5m$, we obtain $m \gtrsim 2$, i.e. even one long-trapped trajectory with the ballistic dynamics can influence the values of high moments. Similar situations were described in Afanasiev et al. (1991) where we called a bundle of the long-staying together trajectories as stochastic jets.

10. Dynamical traps and kinetics

10.1. General comments

In the previous section we described different properties of chaotic dynamics and fractional kinetic equations. This section is about how to make a connection between dynamics and kinetics. The main important feature of the dynamics is its non-uniform process of mixing in phase space, which creates different types of dynamical traps. We will consider the traps in some generalized sense. A trap can be localized in a domain A of phase space and the domain can be bounded by hypersurfaces. For example, a narrow strip around an island can represent two types of trappings: rotation around the island or spiraling along a tube for which the strip is the tube's section. Both types of trappings will be called *flights*. Both types of trappings represent only parts of the trajectories that are trapped into the corresponding domain A or in a cylinder with the A as its cross-section. There exists a connection between the area of an island S and the length of a corresponding flight (Zaslavsky, 1994a, b)

$$\lambda_S = \lambda_\Gamma = 1/\lambda_\ell^2 \tag{283}$$

that follows from the Liouville theorem. Here $\lambda_S, \lambda_\Gamma, \lambda_\ell$ are scaling parameters and (283) refers to the case when the scaling works. Just these scaling constants define the fractal properties of the dynamics through the distributions of Poincaré recurrences and escape time (Zaslavsky, 1994a, b; Afraimovich and Zaslavsky, 1997).

The fractional kinetics is defined by FKE and its characteristic exponents α and β. In this section we discuss how the connection between the kinetic exponents (α, β) and the scaling constants $(\lambda_\ell, \lambda_S, \lambda_\Gamma)$ appears.

10.2. Kinetics and transport exponents

Let us start from the HIT. The corresponding connection was found in Zaslavsky (1994a, b) using the RG approach. Here a qualitative version of this derivation will be presented.

Consider a phase volume Γ_n that corresponds to an island of the nth generation, and let the domain A corresponds to the chain of islands of the nth generation. An integral probability to return to the island boundary, in correspondence to (21), should be

$$P_{\rm rec}^{\rm int}(t_n) \sim \frac{\text{const.}}{t_n^{\gamma-1}} \cdot T_n \ , \tag{284}$$

where the factor T_n is simply the number of islands in the nth generation chain. It is important that we do not consider an arbitrary values of t_n, but, instead, only those that correspond to the hierarchy $\{T_0, T_1, \ldots, T_n, \ldots\}$. Then (284) can be rewritten as

$$P_{\rm rec}^{\rm int}(T_n) \sim \frac{\text{const.}}{T_n^{\gamma-2}} \ . \tag{285}$$

From the other side, this probability should be proportional to the phase volume Γ_n occupied by an island, i.e.

$$P_{\rm rec}^{\rm int}(T_n) \sim \Gamma_n/\Gamma \ , \tag{286}$$

where Γ is the full phase volume and is used for the normalization.

Before continuing the calculations, let us make a comment about the number of islands in the trap. Let us consider a sequence of integers $\{m_1, m_2, \ldots, m_{n_0}\}$ that represents the sequence of the consequent island chains. Then the number of islands at the n_0th hierarchical level is

$$N_{n_0} = m_1 \cdot m_2 \ldots m_{n_0} \tag{287}$$

or, for example, for all equal $m_j = m, N_{n_0} = m^{n_0}$. At the same time, the integers m_n have a meaning of the periods on the nth level of hierarchy, normalized to the basic period T, i.e.

$$m_n = T_n/T \tag{288}$$

and thus $t_n \sim T_n \sim m_n T$. For the case of equal m_j

$$N_{n_0} = m^{n_0} = \lambda_T^{n_0} \tag{289}$$

in correspondence to the HIT.

Making the right-hand sides of (285) and (286) equal and considering the limit $n \to \infty$, we obtain

$$T_n^{\gamma-2} = \text{const.}/\Gamma_n \tag{290}$$

or after substitution of (273) and (283) in (290)

$$\gamma = 2 + \frac{|\ln \lambda_\Gamma|}{\ln \lambda_T} = 2 + \frac{|\ln \lambda_S|}{\ln \lambda_T} = 2 + 2\frac{\ln \lambda_\ell}{\ln \lambda_T} \ . \tag{291}$$

Let us rewrite this expression as

$$\gamma = 2 + \mu \tag{292}$$

with a notation

$$\mu = |\ln \lambda_\Gamma|/\ln \lambda_T = \frac{|\ln \lambda_S|}{\ln \lambda_T} = 2\,\frac{\ln \lambda_\ell}{\ln \lambda_T}\ . \tag{293}$$

The last step is to come back to the FKE (134) for which the transport equation (139) defines the moments

$$\langle |x|^\alpha \rangle = \text{const. } t^\beta \tag{294}$$

or

$$\langle |x| \rangle \sim t^{\mu/2} = t^{\beta/\alpha}\ . \tag{295}$$

The RGK in Section 8.1 provides the final expression (see (244) and (247)):

$$\mu = 2\beta/\alpha = 2\ln \lambda_\ell/\ln \lambda_T \tag{296}$$

which links the transport to the HIT scaling properties.

This result is not universal, for different values of the control parameter can yield different values of α, β and μ (see more in Benkadda et al., 1997a, b, 1999; White et al., 1998; Panoiu, 2000; Kovalyov, 2000). Nevertheless, we can consider different classes of universality (Zaslavsky and Edelman, 2000). For example, for a not too long time one can apply the so-called one-flight approximation for which

$$\mu = 4 - \gamma \tag{297}$$

(Karney, 1983) that will be considered in Section 10.6.

For the case of the net traps, one can estimate

$$\gamma = 1 + \mu = 1 + \frac{2\ln \lambda_\ell}{\ln \lambda_T} \tag{298}$$

(White et al., 1998; Zaslavsky and Edelman, 2000).

In spite of all the three different connections (291) (297) and (298) between the critical exponents that have been found in theory and in numerous simulations (see Sections 12–14), we should make a special discussion of use of these formulas. It will be done at the end of the Section 10.4. Accelerator mode islands were associated with the anomalous transport in many publications (Karney, 1983; Aizawa et al., 1989; Ishizaki et al., 1991).

10.3. Estimates of the exponents

Let us start from the recurrence exponent γ and provide a qualitative algorithm for its estimates. Consider a small initial volume Γ_0 and its time evolution. The enveloped coarse-grained volume $\bar{\Gamma}(t) \geqslant \Gamma_0$ (Gibbs theorem), although $\Gamma(t) = \Gamma_0$ (Liouville theorem). This links the problem of recurrences to the problem of evolution of $\bar{\Gamma}(t)$. One can consider a small ball of the size ϵ that moves in phase space and cover it with time. There are different possibilities:

$$\bar{\Gamma}_1(t) = \text{const.} \cdot t^{1+\delta_1}\ ,$$

$$\bar{\Gamma}_2(t) = \text{const.} \cdot t^{3/2\pm\delta_2}\ ,$$

$$\bar{\Gamma}_3(t) = \text{const.} \cdot t^{2\pm\delta_3}\ ,$$

$$\bar{\Gamma}_4(t) = \text{const.} \cdot t^{3\pm\delta_4}\ , \tag{299}$$

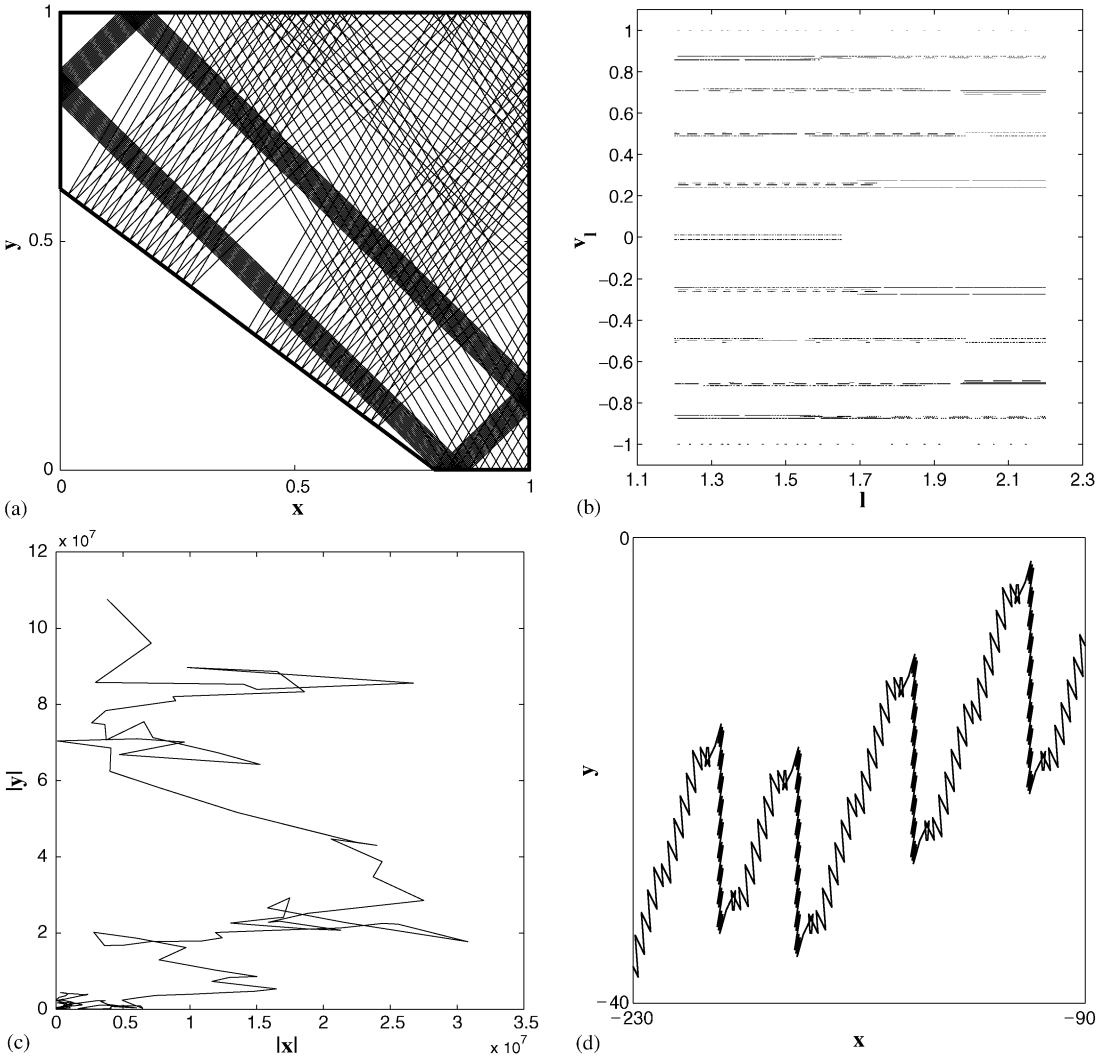

Fig. 15. Rhombic billiard (it is shown only 1/4 part of the rhomb in a square; see also Figs. 15.1d and 15.2): (a) a trajectory in the billiard table shows slow evolution; (b) phase plane ℓ, v_ℓ (ℓ is the upper side of the square); it shows slow evolution along v_ℓ; (c) the same trajectory in the cover plane; (d) a zoom of a piece of the trajectory in (c).

where δ_j are considered as corrections to four different main values. The case of $\bar{\Gamma}_1(t)$ corresponds to an almost one-dimensional coarse-grained dynamics, i.e. almost ballistic wandering. It seems that this case happens in the rhombic and triangular billiards and Fig. 15 confirms this statement (Zaslavsky and Edelman, 2002). The same case can also appear in one-dimensional dynamics (see Young (1999) and some rigorous results therein).

The case of $\bar{\Gamma}_2(t)$ corresponds to the linear evolution along one coordinate and the diffusional type evolution along the other coordinate along which the phase volume growth is proportional to $t^{1/2}$ as in the Gaussian diffusion. This is one of the most typical cases for many systems with

intermittent chaotic dynamics (Zaslavsky and Edelman, 2000; see also Section 10.5). The third case of $\bar{\Gamma}_3(t)$ corresponds to the almost linear phase space coverage along each of two coordinates. The alternations of the coordinates can be randomly distributed but the result provides almost t^2-growth of $\bar{\Gamma}_3(t)$. It seems that it is the case of the original Sinai billiard when $\delta_3 = 0$ (Chernov and Young, 2000). The case of $\bar{\Gamma}_4(t)$ can appear in special situations of 1 1/2 degrees of freedom or in higher dimension system (Rakhlin, 2000). All the power laws in (299) are up to a logarithmic factor that cannot be captured by this type of estimates.

The density distribution of recurrences can be obtained from (299) by applying the formula

$$P_{\mathrm{rec}}^{\mathrm{int}}(t) = \int_t^{\infty} \mathrm{d}t' P_{\mathrm{rec}}(t') = \frac{\Gamma_0}{\bar{\Gamma}(t)} \tag{300}$$

that shows a probability of non-return earlier than after time t, or

$$P_{\mathrm{rec}}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\Gamma_0}{\bar{\Gamma}(t)} = \mathrm{const.}/t^{\gamma_j} \tag{301}$$

with

$$\gamma_j = \begin{cases} 2 + \delta_1 \\ 5/2 \pm \delta_2 \\ 3 \pm \delta_3 \\ 4 \pm \delta_4 \end{cases} \qquad (\delta_j > 0) \tag{302}$$

with some corrections δ_j to the main exponents. Sometimes the corrections cannot be written in the way (302) like it is for the Sinai billiard where the logarithmic multiplier appears.

10.4. One-flight approximation

One of the first approximations for the anomalous diffusion in a map model, similar to the web map, was proposed in Karney (1983). From this work, the transport exponent is

$$\mu = 3 - \beta\ , \tag{303}$$

where β is the exit time exponent, or

$$\gamma = 4 - \mu\ , \tag{304}$$

where γ is the recurrence exponent (compare to (297)). There are different modifications of this derivation related to the same or other models, and each of the modifications has its own initial formulation of the problem with corresponding restrictions (Chirikov and Shepelyansky, 1984; Geisel et al., 1987a, b; Ichikawa et al., 1987; Klafter et al., 1987).

Let x be a variable of the interest (coordinate or momentum) and we are interested in

$$\langle x^2(t)\rangle = \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \langle \dot{x}(t_1)\dot{x}(t_2)\rangle = 2t \int_0^t \mathrm{d}t_1 C(t_1) \tag{305}$$

with the velocity correlation function

$$C(t) = \langle v(t)v(0)\rangle, \tag{306}$$

and the assumption of stationarity of $C(t)$. For the algebraic decay of $C(t)$,

$$C(t) \sim C_0/t^{\gamma_c}, \tag{307}$$

one has

$$\langle x^2(t)\rangle \sim t^{2-\gamma_c} \quad (\gamma_c < 2). \tag{308}$$

In the definition of γ_c, a one-flight approximation means that both $v(0)$ and $v(t)$ in (306) belong to the same flight. Then $C(t)$ is proportional to the probability that a particle stay in the same flight, at least during the time interval t. For the probability density $\psi(t)$ to escape from a flight at time instant t, the probability to escape from the flight at any time instant $t_2 > t_1$ is

$$P_{\text{esc}}(t_2 > t_1) = \int_{t_1}^{\infty} \mathrm{d}t_2 \psi(t_2) \ , \tag{309}$$

and the probability that the escape time t_1 is within the interval (t, ∞) is

$$\begin{aligned} P_{\text{esc}}(t_1 \in (t,\infty)) &= \int_t^{\infty} \mathrm{d}t_1 P_{\text{esc}}(t_2 > t_1 > t) \\ &= \int_t^{\infty} \mathrm{d}t_1 \int_{t_1}^{\infty} \mathrm{d}t_2 \psi(t_2) \ , \end{aligned} \tag{310}$$

which is just $C(t)$ in the one-flight approximation. Combining (307), (308), (310) and (221), we obtain (303) or (304), since it follows from (310) that

$$\gamma - 2 = \gamma_c \ . \tag{311}$$

The main restriction for this result is that the internal structure of the flight (or a trap) is not considered, i.e., the existence of subflights due to stickiness to subislands or a more complicated space–time structure of the singular zone is ignored (see more discussions in Benkadda et al., 1999). Nevertheless, formula (304) can be applied to numerous situations when, for a restricted time window, the one-flight approximation is sufficient.

One more important comment exists for expression (311). It links in a simple way the second moment (305) and the recurrences (or escapes) as it follows from (310) and thereafter. This is also a kind of the acceptance of the one-flight approximation and it is not correct for general situations. Indeed, as it was clear from the CTRW (see Section 6.2) or from FKE (see Section 5.5), at least two independent probabilities define the transport, i.e. the evolution of the moments, while formula (304) expresses the transport exponent through the exit time distribution only.

While expressions for (292) and (298) are qualitatively similar, the one-flight approximation formula (304) is very different since μ appears with different sign. Derivation of (304) uses only exit time distribution without taking into account fractal space properties of a singular zone. Actually, the diffusion process is characterized by two exponents (α, β) and the same for the coarse-grained phase volume and recurrence time exponent γ. The fine structure of singular zones cannot be described by the only flight and correlation between different flights should not be neglected. That is why (292) and (298) are defined by two scaling parameters λ_ℓ, λ_T. There is no smooth transition between, say (298), and (304) since the characteristic time and length do not exist. Nevertheless, there is convergence to $\mu = 3/2$ of the two expressions when $\gamma = 5/2$. This interesting case will be considered in the following section.

10.5. The 3/2 law of transport

The value of $\mu = \frac{3}{2}$ can appear simultaneously with $\gamma = \frac{5}{2}$ in different cases. This remarkable value exists not only for the fractal kinetics (see an example in Venkatarami et al. (1997) when the boundary layer plays an important role). In this section, we speculate on the origin of the exponents,

$$\gamma = 5/2, \quad \mu = \gamma - 1 = \frac{3}{2} \ , \tag{312}$$

based on the FK induced by sticky islands.

Let us assume that the boundary of a sticky island is close to the destroyed island separatrix. Then, a trajectory near the boundary can spend a long time if it is trapped into a regime close to the ballistic one. An effective Hamiltonian of a ballistic motion island is (compare to (270))

$$H_{\rm ef} = a_1(\Delta\phi)^2 + a_2\Delta h - a_3(\Delta h)^3 \ , \tag{313}$$

where $\Delta\phi, \Delta h$ corresponds to deviations of the phase ϕ and energy h values on the trajectory from their special values ϕ^*, h^* on the ballistic trajectory when the island is born due to some bifurcation. Similar Hamiltonians appear for an accelerator mode island (compare to (262)), and the only difference between these two problems is in the value of variables $\Delta\phi, \Delta h$, coefficients a_j, and their physical meaning.

The Hamiltonian structure in (313) is enough to estimate an exit time distribution due to an intermittent escape from the near-separatrix boundary layer of the island. Indeed, the phase volume of the escaping particles is

$$\delta\Gamma = \delta\phi \cdot \delta h \ , \tag{314}$$

where $\delta\phi, \delta h$ are deviations of $\Delta\phi, \Delta h$ from their values on the island border. From Hamiltonian (313) one can obtain for the running away trajectories that $\delta\phi \sim \delta\phi_{\rm max} \sim \delta h^{3/2}$, i.e., $\delta\Gamma \sim \delta h^{5/2}$. Escaping from the boundary layer means growth of the "radial" variable δh with time, which can be put as $\delta h \sim t$, at least for an initial finite time interval. Then from (314)

$$\delta\Gamma(t) \sim t^{5/2} \ . \tag{315}$$

The escape probability from the boundary layer can be estimated as

$$\psi(t) \propto 1/\delta\Gamma(t) \sim t^{-5/2} \tag{316}$$

or

$$\gamma = 5/2 \ . \tag{317}$$

For symmetric flights, applying $\gamma = \mu + 1$ (see (298)) we obtain

$$\mu = 3/2 \ . \tag{318}$$

As it was mentioned above, it is not the only way to get exponents (316) and (317), and there are other possibilities and corresponding restrictions. It is also interesting to comment that formula (304) gives the same result for μ if $\gamma = 5/2$, although this formula has a different region of validity.

10.6. Do the critical exponents exist?

Numerous simulations can present high accuracy results for fairly simple models of chaotic dynamics. Since a rigorous theory of chaotic transport does not exist yet, we can address the question at the title of this section the simulations and, sometimes, to the experiments. It seems that in a rigorous sense, the correct answer should be "not", although for some specific values of the control parameters a unique set of the exponents (α, β, γ) for the FK is very close to the reality. A reason for the negative answer is that general chaotic dynamics is multifractal with a number of different singularities in phase space and with different types of dynamical traps. Even for the case of existence of the only hierarchy of a set of islands with fixed values of the exponents (α, β, γ), the log-periodicity imposes non-linear dependence of the log $\langle |x| \rangle$ vs. log t

$$\log\langle |x| \rangle / \log t \neq \text{const.} \tag{319}$$

As a result, the moment in (319) has a different level of "wiggling", making the left-hand-side of (319) non-constant.

A typical decoy to work with such a type of data is to use an "average" best-fit line. This procedure reveals different results obtained for even slightly different conditions that are of the curves $\log \langle |x| \rangle$ vs. $\log t$ hardly to compare with high accuracy. The "wigglings" are natural and they are not due to fluctuations that can be averaged out. Deviations from (319) or differences between differently averaged curves can be due to different conditions of the simulations, different space–time windows, different procedures of smoothing the curves (for example, smoothing $\langle |x| \rangle$ vs. t is not the same as smoothing $\log \langle |x| \rangle$ vs. t or $\log \langle |x| \rangle$ vs. $\log t$), and so on.

An important way of comparing the results obtained in different ways should include different qualitative features of the investigated model, admitting a possibility of small deviations of the exponent values. There are more serious obstacles in collecting reliable data for the critical exponents, which will be discussed at the end of the following section.

10.7. Straight definition of FKE exponents

Accepting a convenient way of averaging the data, one can collect the values of α and β for the FKE in a straightforward way. For example, the probability density for long steps (flights) can be found in the form

$$W(\ell) \sim \text{const.}/\ell^{1+\alpha} \tag{320}$$

and their time distribution is

$$\psi(t) \sim \text{const.}/t^{1+\beta} \ . \tag{321}$$

Then, the FKE can be applied with the corresponding values of α and β from (320) to (321). This approach proves to be useful in different applications, when the topological nature of FK is not explicit (see for example the running sandpile analysis in Carreras et al., 1999). This type of approach to FK is very general and it can be applied to different kinds of systems, not necessarily the Hamiltonian ones.

11. Dynamical traps and statistical laws

A significant part of our work with statistical data and with dynamical strongly chaotic systems is based on a silent assumption that typical statistical laws and typical principles of the ergodic theory can be applied. In fact, dynamical chaos with complicated phase space pattern reveals a different class of processes and distributions which are non-Gaussian, non-ergodic, non-mixing and many other "nons", if one would apply rigorous definitions of all of these notions (see a review discussion in Zaslavsky, 1999). In this section we discuss how the presence of dynamical traps influences the basic statistical laws and, particularly, how it is related to FK.

11.1. Ergodicity and stickiness

The idea that Poincaré used to prove the theorem of recurrences is fairly simple and it is based on the phase volume preservation for Hamiltonian systems. We would like to give a brief reminder of it, as this will help in understanding of possible consequences of stickiness. Consider a small area of the phase space A with volume Γ_A. For simplicity we can say that within A there is always an area A_1 with volume Γ_{A_1} consisting of particles that leave A within a limited time interval t_1, since otherwise it would not have been necessary to prove the return of particles. Similar reasoning can be used in considering the exit of particles from the area A_1 which should occur within a certain time interval t_2. Otherwise A_1 is sink forbidden by the Liouville theorem of the phase volume preservation. Hence the particles assume a new position, i.e. the area A_2 with volume $\Gamma_{A_2} = \Gamma_{A_1} \neq 0$ and $A_1 \cap A_2 = 0$, since otherwise some particles accumulate in A_1. This consideration can be continued. If the particles, performing a finite motion, never return to the initial area A, then $\Gamma_{A_1} + \Gamma_{A_2} + \cdots = \infty$ because all Γ_{A_i} are equal, which contradicts the condition of the confinement of motion. Hence, the particle must return to A in a finite time T.

The above reasoning demonstrates that a return time cannot be less than an exit time from the area A which can be measured relatively easy. Let $\psi(\tau)\,\mathrm{d}\tau$ be the probability that a typical particle exits the area A in a time τ which falls within an interval $(\tau, \tau + \mathrm{d}\tau)$. Then the probability of the particle exiting the area A at the time $\leqslant t$ is

$$\Phi_e(t) = \int_0^t \psi(\tau)\,\mathrm{d}\tau, \quad \Phi_e(\infty) = 1 \tag{322}$$

(escape probability); and the probability to exit during the time $\geqslant t$, or survival probability, is

$$\Phi_s(t) = \int_t^\infty \psi(\tau)\,\mathrm{d}\tau = 1 - \int_0^t \psi(\tau)\,\mathrm{d}\tau = 1 - \Phi_e(t), \quad \Phi_s(\infty) = 0\ . \tag{323}$$

For the Anosov-type systems, the escape probability density is the same as for the distribution of recurrences (compare to (2.3.5) and (2.5.4)), i.e.

$$\psi(t) = (1/\tau_{\mathrm{rec}}) \exp(-t/\tau_{\mathrm{rec}})\ . \tag{324}$$

Typical distribution of recurrences (and escapes) has the exponential behavior of the (324) type only for a finite time and then it follows an algebraic or another sub-exponential law. An example of $P_{\mathrm{rec}}(t)$ is shown in Fig. 16 for the Sinai billiard. The stickiness appears to the domain of bouncing trajectories (Fig. 17) that has zero measure (!) in phase space. It was found that the asymptotics

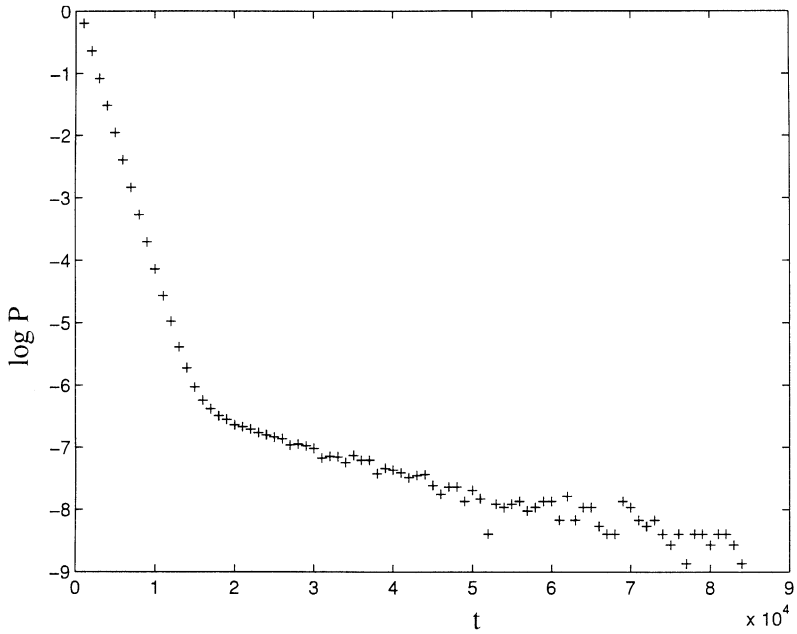

Fig. 16. Distribution density of the Poincaré recurrences for the Sinai billiard with infinite horizon (Zaslavsky and Edelman, 1997).

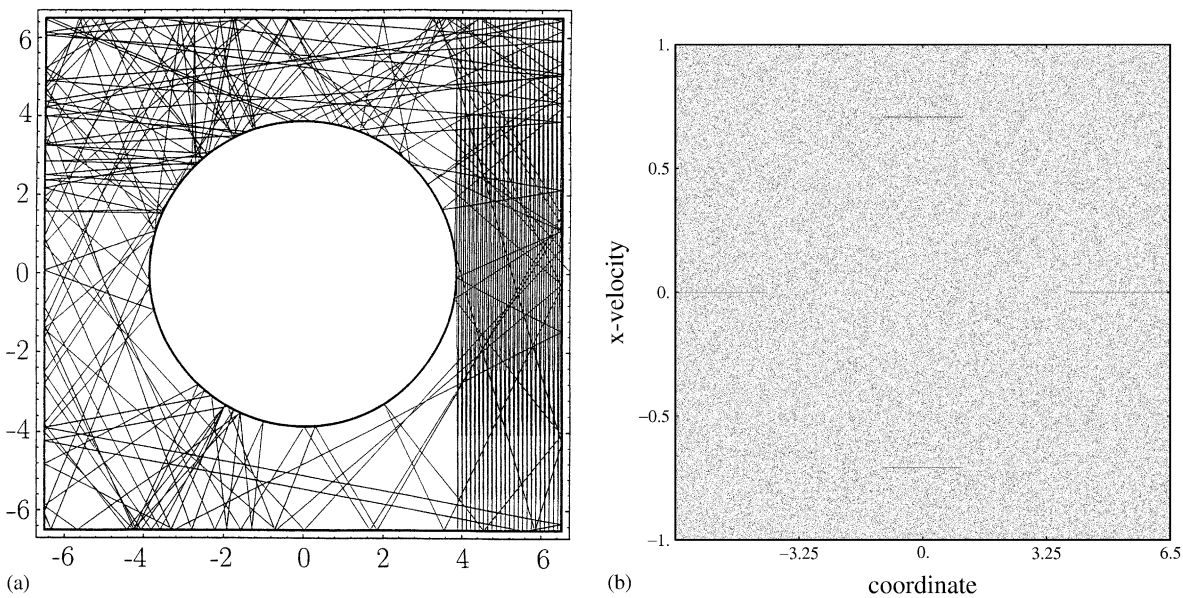

Fig. 17. A sticky part of the trajectory in the Sinai billiard, that bounces between two walls without scatterings (a), and its Poincaré map (b).

of $P_{\rm rec}(t)$ is

$$P_{\rm rec}(t) \sim 1/t^3 \tag{325}$$

up to a possible logarithmic factor (Machta and Zwanzig, 1983; Zacherl et al., 1986; see also rigorous results on the correlation decay in Young, 1999).

Since the finiteness of the mean recurrence time $\tau_{\rm rec}$ (see (17)) for Hamiltonian dynamics, and equality (29), we have a similar restriction for the mean exit time

$$\tau_{\rm exit} = \int_0^\infty t\psi(t)\,{\rm d}t < \infty \tag{326}$$

or, assuming $\psi(t)$ in form (321), this yields

$$\beta > 1\ . \tag{327}$$

Just this restriction as well as the condition $\tau_{\rm rec} < \infty$ leads to an unusual "thermodynamics" for the system with dynamical chaos, which will be discussed in Section 11.4.

Let us make an important comment: Assume that

$$\beta = 1 + \delta, \quad 0 < \delta \ll 1\ . \tag{328}$$

Condition (326) works but the tail of the distributions $P_{\rm esc}(t)$ and $P_{\rm rec}(t)$ is extremely long so that second and higher moments of $P_{\rm rec}(t)$, $P_{\rm esc}(t)$ are infinite. One can expect in such systems arbitrary strong and long-lasting fluctuations as it, for example, happens due to the dynamical trap of the stochastic layer type.

11.2. Pseudoergodicity

For a dynamical variable $x = x(x_0; t)$ with initial condition x_0 and Hamiltonian dynamical equations $x(x_0; t) = \hat{T}(t, t_0)x_0$ with an evolution operator $\hat{T}(t, t_0)$, the ergodicity means that

$$\overline{f(x)} \equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t f(x(x_0,\tau))\,{\rm d}\tau = \langle f(x)\rangle\ , \tag{329}$$

where $\langle\ldots\rangle$ means ensemble averaging over the invariant measure (distribution function) $\rho(x)$ which in typical cases can be reduced to the averaging over $\rho(x_0)$ or simply

$$\langle f(x)\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} f(x_0^{(j)}) \tag{330}$$

and N initial conditions $x_0^{(j)}$ are taken fairly arbitrary, but "well representatively". The function $f(x)$ should be integrable. In practice, we never have a case of being able to perform the limits $t \to \infty$ and $N \to \infty$ but we always can consider fairly large $t_{\rm max}$ and $N_{\rm max}$ restricted by the experimental or computational resources. We never had a problem before in observing deviations from (329) and (330) for well mixing systems. It was always that fairly satisfactory values of $(t_{\rm max}, N_{\rm max})$ could be achieved in simulations.

Although the standard formulation of the ergodic theorem (329) was never considered to fail for more or less typical chaotic systems, we would like to comment that *it can be that equality* (329) *is never achieved* for some special values of the control parameter, in the sense that the deviation

$$|\overline{f(x)} - \langle f(x)\rangle| \equiv \delta(N_{\rm max}, T_{\rm max}) \tag{331}$$

does not approach zero uniformly with respect to the values $(N_{\max}, T_{\max})$, and large and arbitrary long bursts are appearing. This property we call *pseudoergodicity* (see also discussions in Zaslavsky, 1995; Zaslavsky and Edelman, 1997; Afanasiev et al., 1991). Here we would like to formulate this property more accurately using the notions of the dynamical traps introduced earlier.

On the basis of the existence of the dynamical traps, we can propose the following:

Conjecture. *Consider the standard or web maps. For a given arbitrary large T we can indicate such a value $K_{(T)}$ that almost all trajectories from the stochastic sea, being trapped into A, will not escape A during the time interval T counted from an instant t^+ of a trajectory entrance to A till the instant $t^- = t^+ + T$ of the trajectory exit from A. Since the dynamics is area-preserving, such events as the super-long stay in A are infinitely repeated for any trapped at least once trajectory and, moreover, there are infinitely many values of $K_{(T)}$ with such properties* (Zaslavsky, 2002).

If this conjecture is correct, it shows explicitly a "trouble" point of the chaotic dynamics when phase space consists of islands or specific boundaries and dynamical traps. This point should be explained more carefully. The presence of islands in phase space means automatically a kind of non-ergodicity, since any trajectory from stochastic sea can never enter any island, and vice versa. Nevertheless, the ergodic property (329) must be applied only to the part of the phase space, which is stochastic sea. This means the above conjecture is formulated for that part of the phase space.

11.3. Weak mixing

The weak mixing property reflects a more deep side of the non-uniformity of dynamics, comparing to the pseudoergodicity. For square integrable functions $G_1(x), G_2(x)$, the mixing means

$$\lim_{n\to\infty} [\langle G_1(\hat{T}^n x) G_2(x)\rangle - \langle G_1(x)\rangle\langle G_2(x)\rangle] = 0 \ , \tag{332}$$

where we, for a simplicity, consider the discrete evolution of the system

$$(x_{n+1}, t_{n+1}) = \hat{T}^n(x_n, t_n) \tag{333}$$

and $\hat{T}$ is a time-shift operator. Mixing (332) is equivalent to the correlation decay

$$\lim_{n\to\infty} R_n = \langle \hat{T}^n x \cdot x\rangle - \langle x\rangle^2 = 0 \ . \tag{334}$$

The weak mixing means that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} [\langle G_1(\hat{T}^k x) G_2(x)\rangle - \langle G_1(x)\rangle\langle G_2(x)\rangle]^2 = 0 \tag{335}$$

(Cornfeld et al., 1982), i.e. decay of correlations of the functions G_1, G_2 exists as a time average of the square correlator. This property makes a strong difference between systems with mixing and weak mixing. Let $\delta R(t)$ be a fluctuation of the correlation function. For the systems with weak mixing $\delta R(t)$ can be large and can last arbitrary long. The only restriction for $\delta R(t)$ is that such large-long correlations should be fairly rare so that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{d}\tau |\delta R(\tau)|^2 = 0 \ , \tag{336}$$

i.e. probability of large-long fluctuations may not be exponentially small. As it was mentioned before and will be seen in the following section is associated with strongly intermittent behavior of trajectories and long-lasting bursts (flights).

11.4. Maxwell's Demon

The specific type of chaotic dynamics with stickiness, flights, bursts, etc. impose a possibility of the Maxwell's Demon, a conceptual device proposed by Maxwell in 1871 and designated to work against the thermodynamics second law (see a collection of the related works in Leff and Rex, 1990). The device (Demon) is located near a hole in the division between two chambers, and it can distribute the particles, that approach the hole, correspondingly to some rule. For example, the particles with velocities larger than v_0 cannot go through the hole in the direction from chamber 1 to chamber 2.

Maxwell, by himself, considered a possibility of using the Demon for a situation that contradicts the basic thermodynamic principle. It is supposed that the Demon considers each trajectory separately, contrary to the thermodynamic consideration that involves an ensemble of particles as a whole system. It was proposed in Zaslavsky (1995) and Zaslavsky and Edelman (1997) to exploit the stickiness as a ready-made design of the Maxwell's Demon. A typical approach of statistical thermodynamics neglects fluctuations of the exponentially small probability. In the case of FK, a sub-exponential probability of large-long fluctuations can create a non-equilibrium state that does not relax during an arbitrary long time.

Just the described possibility was demonstrated in Zaslavsky and Edelman (1997). Two chambers (Fig. 18) have a small hole in the partition and scatterers in their centers that forms one chamber as the Sinai billiard, and another one as the Cassini billiard (i.e. Cassini oval as a scatterer). A sample of a trajectory in the Cassini billiard periodically continued in both directions (Fig. 19) show flights and traps typical for FK.

Consider a distribution of the recurrences to the partition. The pressure is proportional to the number of recurrences during some time interval, but the distribution of this number does not have a

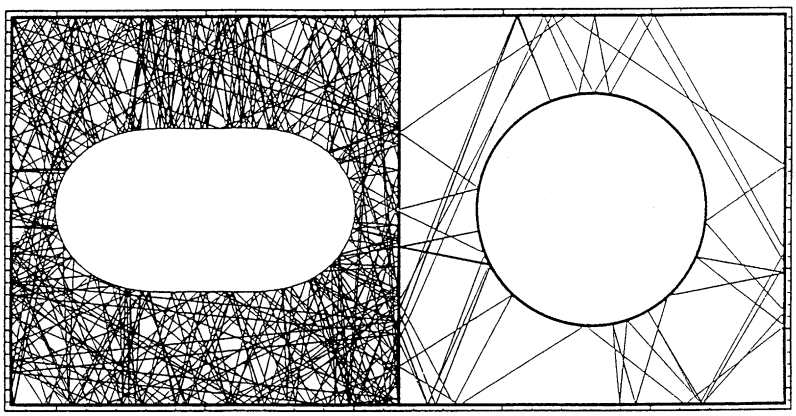

Fig. 18. A sample of a trajectory in the Cassini (left) and Sinai (right) billiards contacting through a hole in the partition (Zaslavsky and Edelman, 1997).

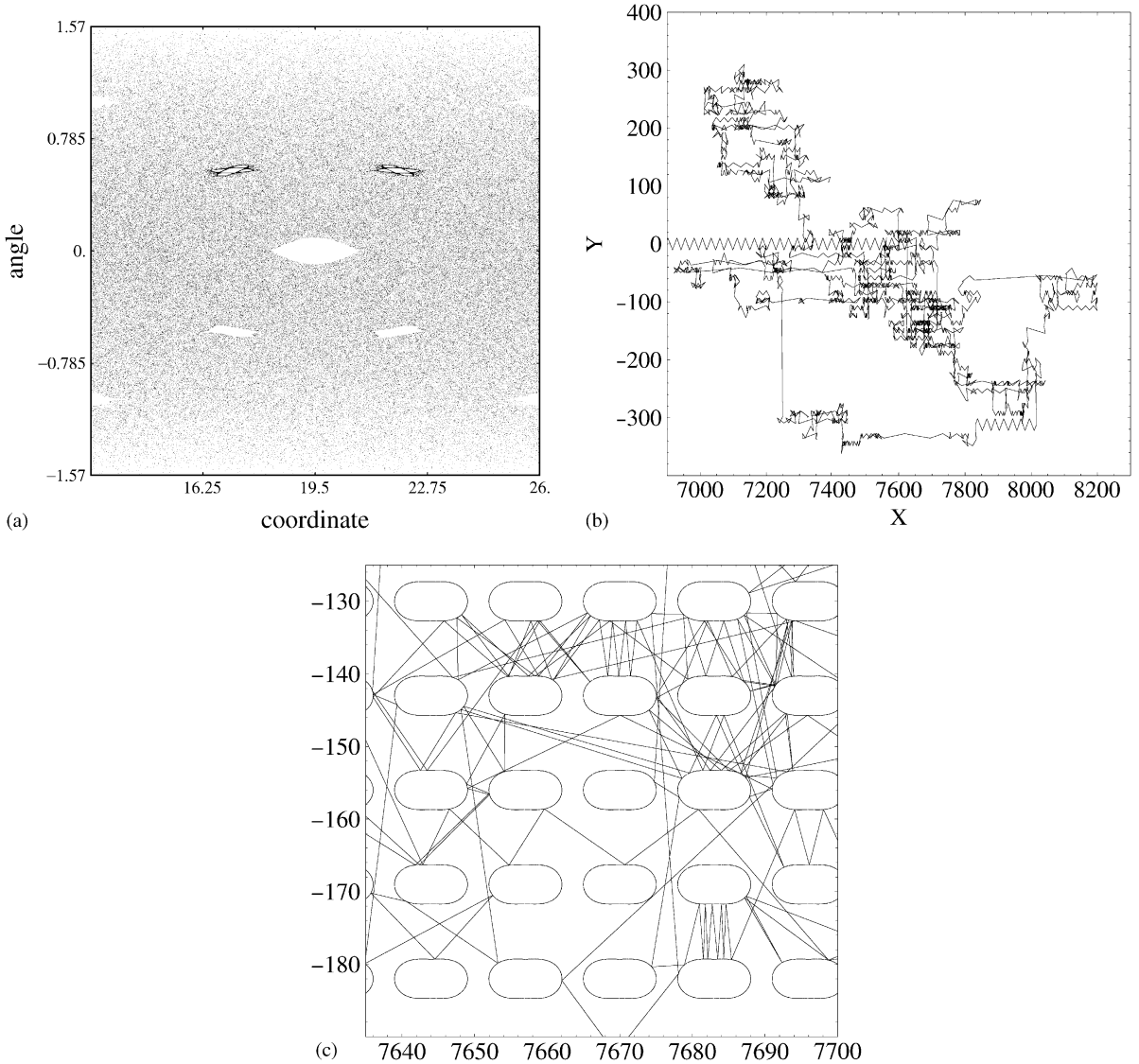

Fig. 19. A sticky trajectory for the Cassini billiard: (a) phase space with the Poincaré section of the trajectory and sticky islands; (b) the same trajectory in the cover space; (c) a zoom of a part of the trajectory (Zaslavsky and Edelman, 1997).

characteristic time and, moreover, exhibits large-long fluctuations. The distribution of the recurrences is different for the left side of the partition and for the right one. This difference does not relax for any finite time and that makes non-equal pressures from two sides of the partition (Fig. 20). The Maxwell's Demon, in this example, is not located at any specific point, but is "distributed" along full billiard as a specific geometry or, better to say, as a specific phase space topology.

This "demonized" feature of Hamiltonian dynamics is a typical one, and it can be observed in different models which do not include the particular in an explicit form. As another example, consider

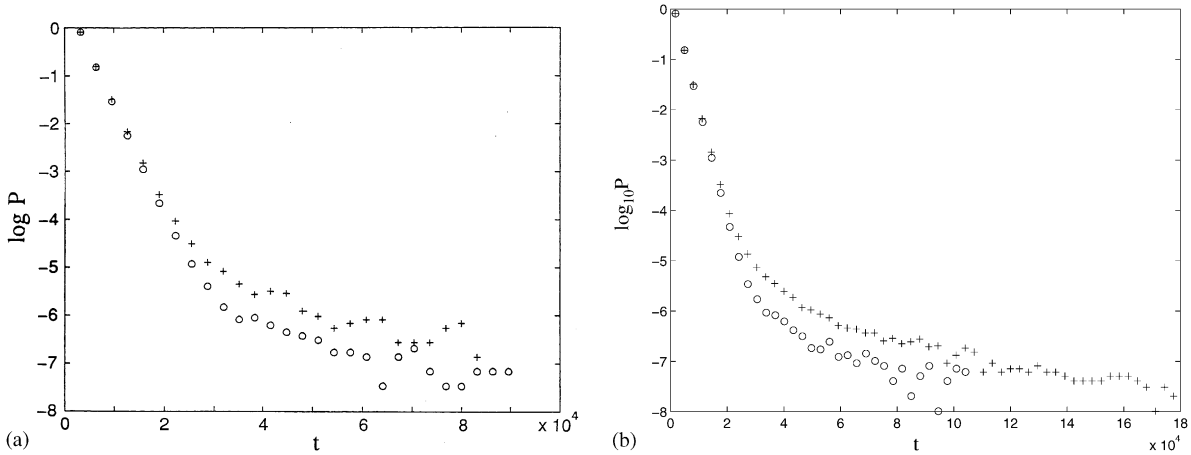

Fig. 20. Distributions of the recurrences for the Sinai billiard part (circles) and the Cassini one (crosses) with the closed hole in the partition (a), and the open hole (b) demonstrates a persistence of the difference during the time 1.16×10^{10} for each of the 37 trajectories (Zaslavsky and Edelman, 1997).

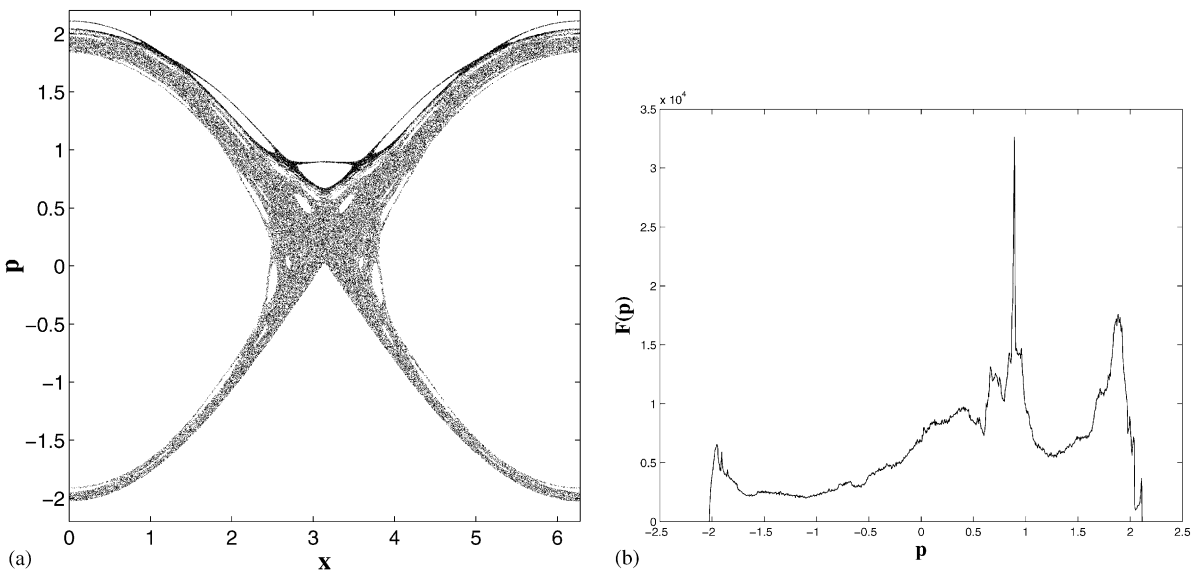

Fig. 21. Stochastic layer trap (dark area in (a)) for the perturbed pendulum model and the corresponding strongly asymmetric distribution of moments $F(p)$ in (b): $\epsilon = 0.181$; $v = 5.4$; $t = 6 \times 10^6 \times 2\pi/v$.

a stochastic layer for the perturbed pendulum:

$$\ddot{x} + \omega_0^2 \sin x = -\epsilon\omega_0^2 \sin(x - vt) \ . \tag{337}$$

In Fig. 21 (Zaslavsky, 1995), we show a sticky trajectory that corresponds to the ballistic mode. It can be captured into a fairly narrow layer for a time of the order 10^7, preventing a relaxation to

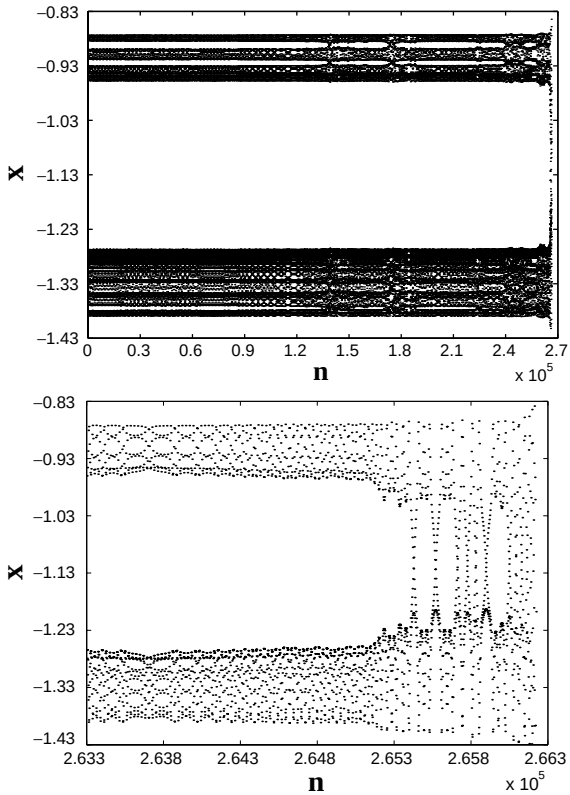

Fig. 22. A long trapped trajectory for about 10^5 iterations of the standard map ($K = 6.908745$) is "cool" (top) until it escapes (bottom).

the uniform distribution of the trajectory in the stochastic sea. As a result, a strongly asymmetric distribution function of momentum $F(p)$ occurs. This type of phenomenon is typically used for stochastic rachets (Mateos, 2000) or for the current generation using the broken symmetry due to a time–space non-uniformity (Flach et al., 2000; Denisov and Flach, 2001). More other features of the chaotic dynamics related to its "sporadic randomness" and a properties of the Maxwell's Demon can be found in Aquino et al. (2001). An experimental realization of the Maxwell's Demon was discussed in Kaplan et al. (2001).

11.5. Dynamical cooling (erasing of chaos)

Long flights of random trajectories can also be considered as the trajectory entrapment into a small domain of phase space. An example in Fig. 21 displays such entrapment and the corresponding phase space domain. It is clear from the figure that being trapped, the trajectory sharply reduces the dispersion along the momentum (or more accurately, along the perpendicular direction to the stochastic jet). More effective entrapment is demonstrated in Fig. 22 for the standard map (Zaslavsky, 2002), which show a kind of regularity of the dynamics within the trapping domain. This property of

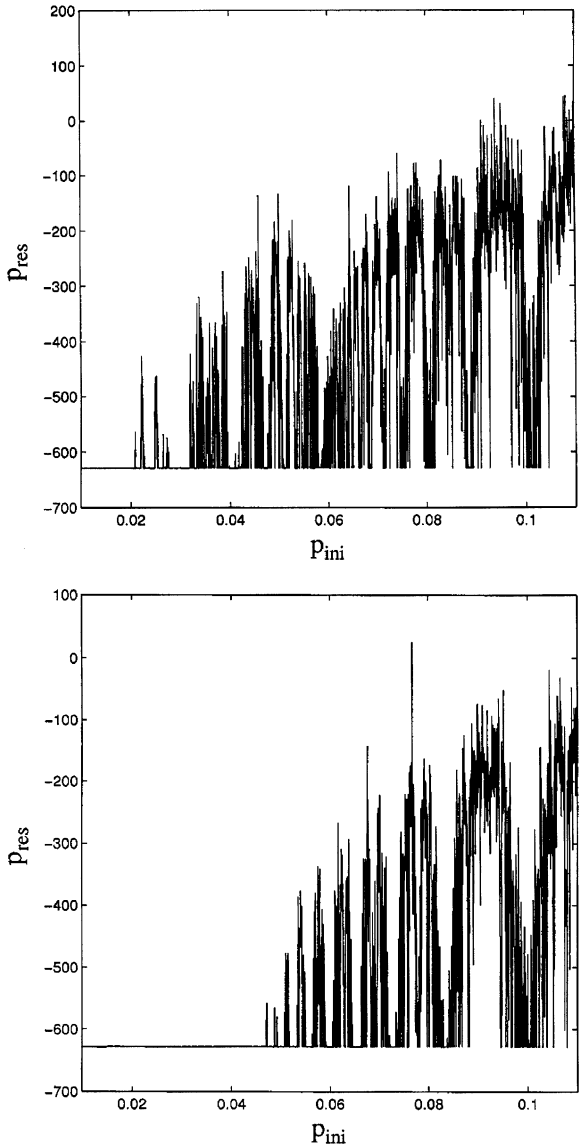

Fig. 23. Chaos erasing demonstration for the standard map: $K = 6.92$ (top) when the transport is normal and the chaos is "normal"; $K = K^* = 6.908745$ when the transport is anomalous due to the stickiness. The momentum at the receiver P_{res} do not appear as a noise in the interval of the initial moments $P_{ini} \in (0.1, 0.45)$ (Zaslavsky and Edelman, 2000).

the chaotic dynamics can be called *dynamical cooling*. The main idea of the cooling (Zaslavsky and Abdullaev, 1995; Zaslavsky and Edelman, 2000) is to tune the control parameter in such a way that creates effective dynamical traps and cool the trajectories for a fairly long time. The effectiveness of this way is demonstrated in Fig. 23 for the standard map. The momentum is "cooled" for a special value of K and chaos is erased in the interval of initial conditions $P_{ini} \epsilon$ (0.02–0.05) where a straight

line means no momentum dispersion. The dispersion, of course, appears if we consider trajectories from this interval for a longer time, which means that the trajectories are leaving the trap after a while. We also can call this phenomenon as *erasing of chaos*.

11.6. Fractal time and erratic time

The concept of *fractal time* has already been mentioned in Sections 7.2 and 9.3. We need to extend its discussion using some notions of Sections 10 and 11. The notion of fractal time was introduced in Berger and Mandelbrot (1963) and Mandelbrot and Van Ness (1968) through a specific, as it was mentioned, Gaussian process $X(t)$ for which

$$\langle [X(t+\tau) - X(t)]^2 \rangle = \text{const.} \cdot \tau^{2\tilde{H}} \tag{338}$$

with the so-called Hurst exponent $\tilde{H} < 1$. The appearance of the fractional powers of time τ in (338) was interpreted as a new type of scaling of time that does not coincide with our "trivial" continuous time scaling. A significant extension of the fractal time notion arrived from the works (Montroll and Scher, 1973; Shlesinger, 1988, 1989; Scher et al., 1991), and especially, with the example of Weierstrass random walk (Hughes et al., 1981, 1982) with a significantly non-Gaussian type of the process. Particularly, an important difference between the Gaussian and WRW processes is that the moments of time are finite in the first case and infinite, starting from some order and for higher orders, for the second case.

Defining a process with events $X_j = X(t_j)$, we also define a set of instants $\{t_j\} = \{t_1, t_2, \ldots\}$ when the events occur. The set $\{t_j\}$ can be of different structure and, particularly, it can possess fractal properties. Dynamical systems present an explicit object where the fractional time concept is a natural and unavoidable notion. The sequence of Poincaré recurrence cycles is, perhaps, the best example of fractal time. In fact, this sequence can be of a more complicated structure, such as multi-fractal (Afraimovich, 1997; Afraimovich and Zaslavsky, 1997).

The fractal time in dynamical systems occurs due to the dynamical traps. For the HIT the fractal time is simply associated with time intervals that a trajectory spend near the islands of different hierarchical levels. In more complicated situations, out of special values of the control parameter, the time sequences can be multi-fractal or even more complicated.

Let us recall the notion of weak mixing in Section 11.3. In a similar way, consider a set of recurrence cycles t_j and their distribution density $P_{\text{rec}}(t)$ normalized as

$$\int_0^\infty P_{\text{rec}}(t)\,\mathrm{d}t = 1, \quad \tau_{\text{rec}} = \langle t \rangle_{\text{rec}} = \int_0^\infty t P_{\text{rec}}(t)\,\mathrm{d}t < \infty \ . \tag{339}$$

In the previous section we consider asymptotic properties of $P_{\text{rec}}(t)$ as

$$P_{\text{rec}}(t) \sim \text{const.}/t^\gamma, \quad t \to \infty \ . \tag{340}$$

In chaotic dynamics, it can be a situation when (340) does not exist but, instead, we have

$$P_{\text{rec}}(t) = \text{const.}/t^\gamma + \delta P(t) \tag{341}$$

with the following properties of $\delta P(t)$.

First, let us rewrite mean recurrence time $\tau_{\rm rec}$ as

$$\tau_{\rm rec} = \lim_{t\to\infty} \frac{1}{N(t)} \sum_{j=1}^{N} t_j = \lim_{t\to\infty} [t/N(t)] \ , \tag{342}$$

where $N(t)$ is a number of recurrences during time t and the summation is performed along one trajectory. Expression (342) follows from the ergodicity. Second, split $N(t)$ as

$$N(t) = N_0(t) + \delta N(t) \ , \tag{343}$$

where $\delta N(t)$ is a number of recurrences distributed as $\delta P(t)$. Then, instead of (342), we have

$$\tau_{\rm rec} = \lim_{t\to\infty} \left\{ \frac{1}{N(t)} \left(\sum_{j\in N_0} t_j + \sum_{j\in\delta N} t_j \right) \right\} = \lim_{t\to\infty} [(t_0(t) + \delta t)/N(t)] \ , \tag{344}$$

where t_0 is full time of all recurrences that are associated with the scaling distribution (340) and δt is the same for $\delta P(t)$.

Assume that the recurrences that are associated with $\delta P(t)$ have no scaling properties, and that

$$\lim_{t\to\infty} \delta t/t_0(t) = 0 \ , \tag{345}$$

i.e. $\tau_{\rm rec}$ and other moments may not depend asymptotically on the recurrences from $\delta P(t)$ but, nevertheless, these types of recurrences can correspond to long-large fluctuations of the dynamical trajectories. We will call this type of time sequence *erratic time*. The erratic time events are not important in evaluation of moments but they are very important for statistical properties of fluctuations. To conclude this subsection, let us mention that the distribution of recurrences and exit times appears in the literature more and more often with respect to different applications, including even such events as the stock market behavior (Baptista and Caldas, 2000; Baptista et al., 2001).

11.7. Collection and interpretation of data

The goal of this section is a discussion of some important properties of the dynamical chaotic systems with respect of their comparison to the experimental and simulation data.

11.7.1. Truncated distributions

It was mentioned in Sections 3.5 that one should consider only truncated distributions which automatically provide all moments to be finite, although not all higher moments can be considered since the truncation. It also follows that a consideration of the asymptotic properties of the truncated distributions

$$\langle |x|^{2m} \rangle_{\rm tr} \sim t^{\mu(m)} \tag{346}$$

is not only a practical convenience, but also a necessity to relate an interpretation of the probabilistic process to the realistic situation. Such truncations were not important for the Gaussian processes but they are necessary for the distributions in chaotic dynamics.

11.7.2. Flights and Lévy Flights

The previous comment was related to the large time asymptotics. Typically, there is a transition time of order 10^3–10^4 from the dynamics that looks chaotic and well mixing to the dynamics with many long flights. A typical distribution of the recurrences, for example, is shown in Fig. 16 where the distribution is exponential and then become the power-wise. This type of behavior is neither Lévy process, nor Lévy-walk. We call these kind of processes as a Lévy type, but we always should have in mind that the application of the rigorous results of the probabilistic theory of Lévy processes (or Lévy walks, CTRW, etc.) can be applied only to the time interval $t_{\min} < t < t_{\max}$ with cuts for both sides of time variables.

11.7.3. Elusive exponents

Two of the above comments require a careful comparison of critical exponents (α, β, γ) as well as their evaluation. This warning should be strengthened by the comments about multi-fractality, log-periodicity, erratic time, and others.

11.7.4. Monte Carlo simulation

Typically, our simulations are considered for fairly long time and for many initial conditions (many trajectories). Due to the pseudoergodicity this strategy should be changed. Long time simulations are desirable but are not effective enough. Much more effective are simulations with a large number of initial conditions.

12. Fractional advection

12.1. Advection equations

The dynamics of particles, advected by a flow $\mathbf{v}(\mathbf{r}, t)$, is described by the equation

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}, t) \ , \tag{347}$$

where the vector field $\mathbf{v}$ is assumed as a given field. Other names for the advected particles are: tracers, passive particles, Lagrangian particles, etc. The set of Eqs. (347) can be considered as a non-linear dynamical system with (typically) chaotic dynamics of particles for some space domains. Several special cases are of the common interest.

(a) *Two-dimensional case*: $\mathbf{r} = (x, y)$. For the incompressible fluid

$$\text{div } \mathbf{v} = 0 \tag{348}$$

and (347) can be written in the form

$$\dot{x} = -\partial\psi(x, y, t)/\partial y, \quad \dot{y} = \partial\psi(x, y, t)/\partial x \tag{349}$$

with a stream function ψ that plays a role of the Hamiltonian of the system.

(b) *Three-dimensional stationary flow*: $\mathbf{r} = (x, y, z)$; $\mathbf{v} = \mathbf{v}(x, y, z)$. For the incompressible case (348), Eqs. (347) describe the streamlines behavior in space, i.e. particles move along streamlines (fieldlines). Eqs. (347) can be rewritten in Hamiltonian form, and they are equivalent to 1 1/2

degrees of freedom system with a chaotic dynamics for general situation (Aref, 1984; Zaslavsky et al., 1991).

(c) *Magnetic chaos*: For any given stationary vector field $\mathbf{B} = \mathbf{B}(x, y, z)$ equations

$$\mathrm{d}x/B_x = \mathrm{d}y/B_y = \mathrm{d}z/B_z \tag{350}$$

describe the fieldlines behavior in space, and can be considered as a dynamical system (see for example Morozov and Soloviev, 1966) with possible chaotic trajectories—the fact well known in plasma physics before the Lagrangian chaos. Eqs. (350) can be rewritten in a similar to (347) form

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \mathbf{B}(\mathbf{r}) \tag{351}$$

with s as a coordinate along a trajectory.

(d) *Beltrami flow*: A particular type of the incompressible stationary flow for which

$$\mathbf{v} = \kappa \operatorname{curl} \mathbf{v} \ . \tag{352}$$

A special type of Beltrami flow was proposed by Arnold (1965), and chaotic streamlines were observed by Henon (1966) when $\kappa = \text{const.}$ (see also Zaslavsky et al., 1991). The Beltrami condition (352) is also important for the magnetic field and related to the so-called force-free field if $\kappa = \text{const.}$:

$$\mathbf{B} = \kappa \operatorname{curl} \mathbf{B} \ . \tag{353}$$

There were many discussions and observations of the anomalous transport of the advected particles or magnetic fieldlines with

$$\langle r^2 \rangle \sim t^{\mu} \tag{354}$$

and $\mu > 1$ (Cardoso and Tabeling, 1988; Solomon and Fogleman, 2001; Solomon and Gollub, 1988; Weiss and Knobloch, 1989; Beloshapkin et al., 1989; Young and Jones, 1990; Chernikov et al., 1990; Aranson et al., 1990; del-Castillo-Negrette and Morrison, 1993; Rom-Kedar et al., 1990; Zaslavsky and Tippett, 1991; Jones and Young, 1994; Artale et al., 1997; Benkadda et al., 1997a, b). To explain the anomalous transport of advected particles, a fractional kinetics was proposed (Zaslavsky et al., 1993). In this section we consider in more detail three special cases.

12.2. Fractional transport by Rossby waves

Rossby waves is one of the most important physical phenomenon in the planetary dynamics of the oceans and atmosphere dynamics (Pedlosky, 1987). The interest to this type of dynamics was extended due to their applications to the Jovian Red Spot and similarity to the drift waves in plasmas (Nezlin, 1994; Sommeria et al., 1989). Recent experiments in the rotational tank show the existence of anomalous superdiffusive transport of passive particles, their Lévy flights and stickiness (Sommeria et al., 1989; Solomon et al., 1993, 1994; Weeks et al., 1996).

A model of the sheared Rossby flow was proposed by del-Castillo-Negrette and Morrison (1993). They use a two-dimensional streamfunction

$$\psi(y,t) = \psi_0(y) + \sum_{j=1}^{N} \psi_j(y) \exp[\mathrm{i}k_j(x - c_j t)] \tag{355}$$

with a radial variable y, phase x, and velocities c_j and wave vectors k_j that should be defined by the boundary conditions of the problem. It was found in the referred experiments of H. Swinney and his group that

$$\langle (\Delta x)^2 \rangle \sim t^{\mu} \tag{356}$$

with $\mu > 1$ (superdiffusion) due to the presence of ballistic type trajectories.

A fairly massive simulation of model (355) was performed by Kovalyov (2000). A typical Poincaré section is shown in Fig. 24 which displays a concentration of a tracer near the boundaries of stochastic sea and islands. The main message from these simulations is the existence of different types of the anomalous transport of (356) depending on the control parameter (amplitude of the perturbation) and time intervals. The values of μ were found within an interval (1.13–1.74). The results show a complicated physical behavior with different regimes depending on the leading mechanism of intermittency.

12.3. Hexagonal Beltrami flow

Beltrami flows with arbitrary q-fold plane symmetry were introduced in Zaslavsky et al. (1988) as solutions to (351) for $\kappa = \text{const.} = 1$. Particularly, it was shown that the dynamics of tracers is defined by a specific phase space topology. More specifically, the real (x, y, z) space possesses three-dimensional web-type connected channels such that the tracer dynamics are chaotic inside the channels, called *stochastic web* (SW). An example of the SW with a plane hexagonal symmetry can be obtained from the flow

$$\mathbf{v} = (-\partial\psi/\partial y + \epsilon \sin z, \partial\psi/\partial x + \epsilon \cos\psi; \psi) \ ,$$

$$\psi = \cos x + \cos((x + \sqrt{3}y)/2) + \cos((x - \sqrt{3})/2) \ . \tag{357}$$

It was found that the advection in flow (357) proves to be anomalous, i.e.

$$\langle R \rangle = \langle (x^2 + y^2)^{1/2} \rangle \sim t^{\mu/2} \tag{358}$$

with μ shown in Fig. 25 (Chernikov et al., 1990; Petrovichev et al., 1990). As it is seen from Fig. 25, the transport exponent $\mu = \mu(\epsilon)$ and deviates strongly from the value $\mu = 1$ at some interval of ϵ. It was also shown that within this interval of values of ϵ there exist a set of the ballistic type islands which appears (and disappears) due to a bifurcation, and that the border of the islands is sticky to the tracers.

The next step of studying the advection in model (357) was performed in Zaslavsky et al. (1993) where a specific value of ϵ was studied that generated a HIT with an approximate self-similarity of the islands area and the period of rotation inside the islands. A FK description was based on the time-dependent generalization of the WRW model.

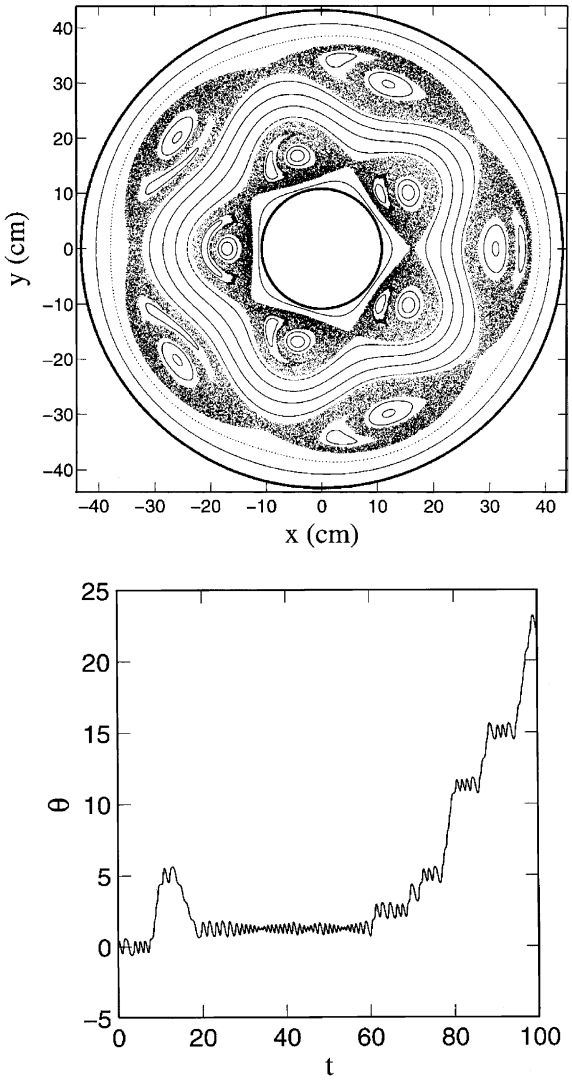

Fig. 24. Poincaré section for the streamlines in the rotational fluid experiment (top) and the angle vs. time (bottom) that shows trappings and flights (Kovalyov, 2000).

Another flow, important for the geophysics, was considered by Agullo et al. (1997) where the web had cylindrical symmetry:

$$v_r = -\frac{1}{r}\frac{\partial\psi}{\partial\phi} + \frac{\epsilon}{r}\sin z \ ,$$

$$v_\phi = \frac{\partial\psi}{\partial r} - \frac{\epsilon}{r}\cos z \ ,$$

$$v_z = \psi \tag{359}$$

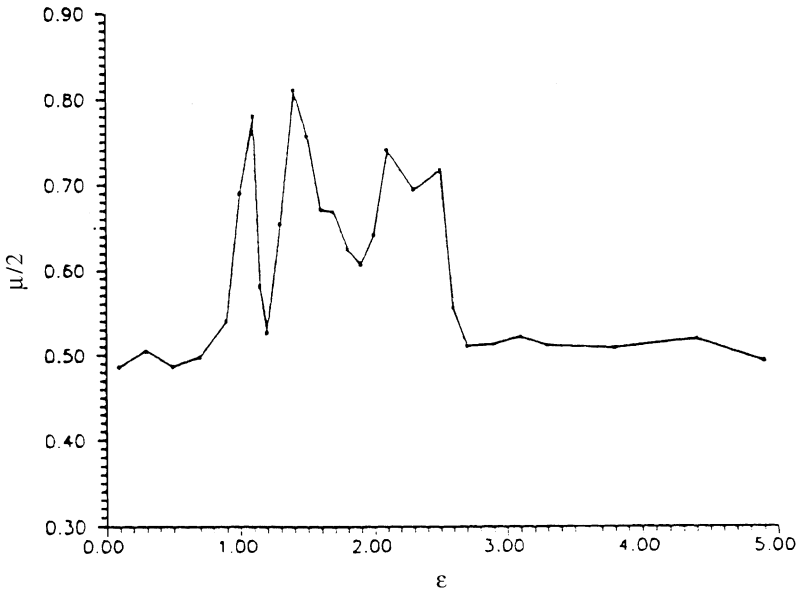

Fig. 25. Transport exponent μ vs. perturbation parameter ϵ for the hexagonal Beltrami flow (Chernikov et al., 1990).

with ψ that satisfies the Helmholtz equation

$$\Delta\psi + \psi = 0 \ . \tag{360}$$

$\psi = J_3(r)\cos 3\phi$ was selected with the Bessel function $J_3(r)$. The tracers were distributed along the stochastic layer displaying the stickiness although the anomalous transport in the helical Beltrami flow was not studied.

12.4. Three point vortices flow

Advection in a system of point vortices is a paradigm of chaotic dynamics in fluids (Aref and Pomphrey, 1980, 1982; Aref, 1984; Ottino, 1989; Crisanti et al., 1992; Babiano et al., 1994; Neufeld and Tel, 1997; Provenzale, 1999; Boatto and Pierrehumbert, 1999; Meleshko, 1994; Zanetti and Franzese, 1993; Weiss et al., 1998; Meleshko and Konstantinov, 2002). The dynamics of the system of point vortices with coordinates $z_k = x_k + \mathrm{i}y_k$, written in a complex form for the convenience, is defined by the stream function

$$\psi(z_1, z_1^*; \ldots; z_N, z_N^*) = -\frac{1}{2\pi} \sum_{j>k\geqslant 1}^{N} \kappa_j \kappa_k \ln |z_j - z_k| \ , \tag{361}$$

which plays a role of the Hamiltonian, and the equations

$$\kappa_k \dot{z}_k = -\mathrm{i}\, \frac{\partial\psi(z_1, \ldots, z_N)}{\partial z_k^*} \ ,$$

$$\kappa_k \dot{z}_k^* = \mathrm{i}\, \frac{\partial\psi(z_1, z_1^*, \ldots, z_N^*)}{\partial z_k} \quad (k = 1, \ldots, N) \tag{362}$$

(Lamb, 1945).

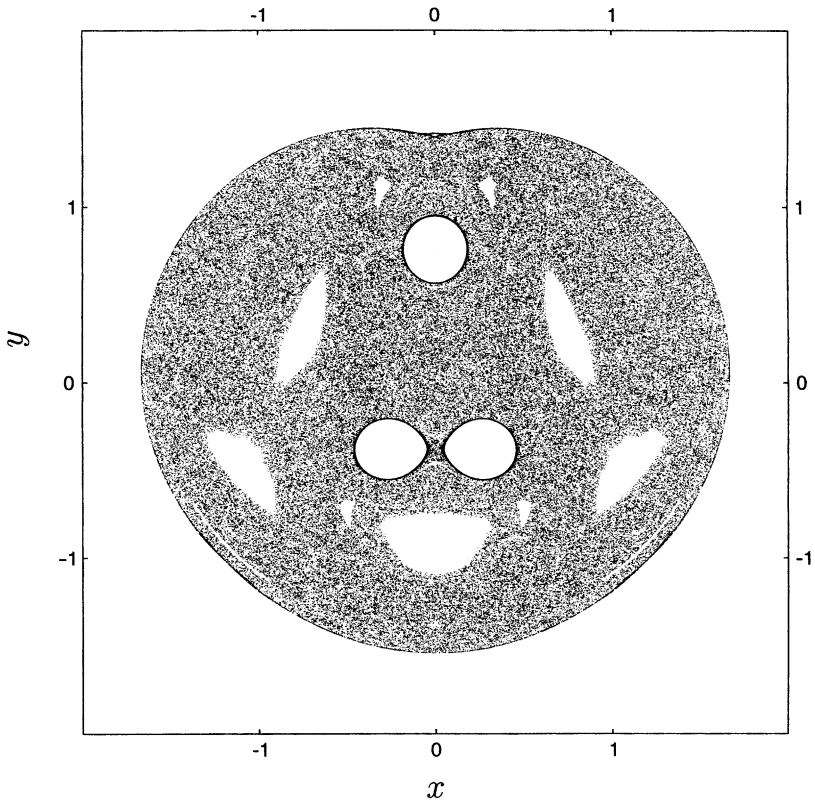

Fig. 26. Poincaré section of a tracer in the 3-vortex flow. There is stickiness around the three cores with the vortices in the centers of the cores, and the stickiness to the outer boundary. Other empty spaces correspond to the resonant islands (Kuznetsov and Zaslavsky, 2000).

Similar are the equations for a tracer (passive particle) with coordinates $z = x + \mathrm{i}y$:

$$\dot{z} = -\mathrm{i}\,\frac{\partial \Psi(z,z^*;t)}{\partial z^*}\ ,$$

$$\dot{z}^* = \mathrm{i}\,\frac{\partial \Psi(z,z^*;t)}{\partial z} \tag{363}$$

with the stream function

$$\Psi(z,z^*;t) = -\frac{1}{2\pi}\sum_{k=1}^{N}\kappa_k \ln|z - z_k(t)|\ . \tag{364}$$

A non-trivial feature of the Eq. (364) is that $z_k(t)$ are solutions of (362) and are supposed to be known. The system of three point vortices is integrable and functions $z_k(t)$ are quasiperiodic (except of the singular solutions), and the system of more than three vortices is non-integrable with chaotic orbits for some phase space domains (Novikov, 1975; Ziglin, 1982). In contrast to the vortices, tracers have chaotic orbits even in the field of three vortices and the tracers transport is, generally

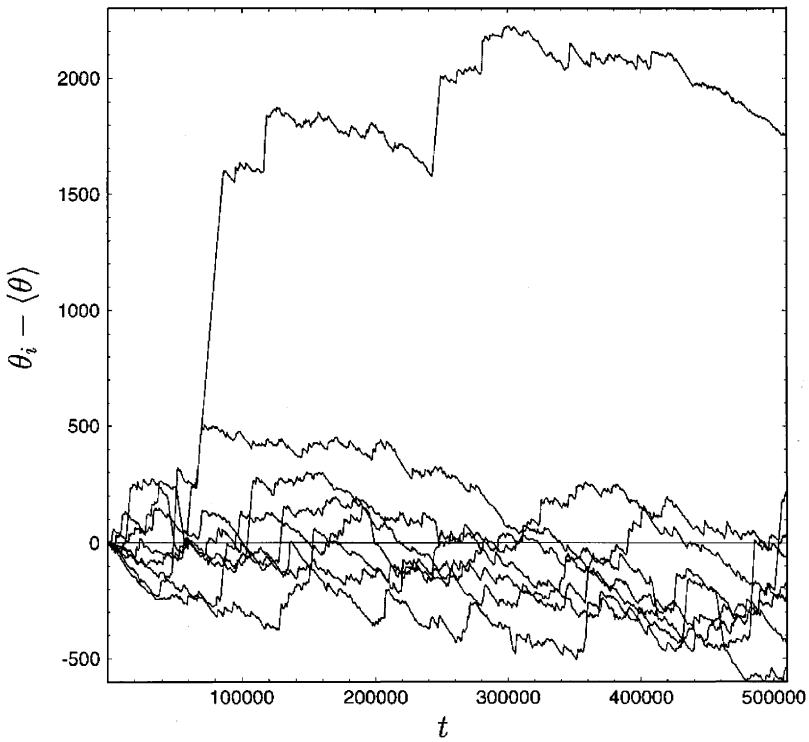

Fig. 27. Rotations of tracers with numerous flights (Kuznetsov and Zaslavsky, 2000).

speaking, anomalous. We will distinguish two cases: all κ_j are of the same sign and collapse is impossible; κ_j have different signs and collapse is possible.

For the case $\kappa_j = 1$ $(\forall j)$, i.e. the collapse is impossible, the problem of anomalous advection was studied in detail in Kuznetsov and Zaslavsky (1998, 2000). A typical phase space with a chaotic sea is shown in Fig. 26. Three almost circular islands, called cores, surround the vortices, and they are filled by KAM invariant curves. The vortices are in the centers of the cores. Other islands are the resonant ones and their presence, number, and shape strongly depend on the control parameters. The cores were discovered in Babiano et al. (1994), and their theory was given in Kuznetsov and Zaslavsky (1998).

The trajectories of tracers are sticky to the island and core boundaries (see Fig. 26 and dark strips around the cores), and to the external boundary of the stochastic sea. The stickiness generates flights (Fig. 27) of an almost monotonic rotation of tracers around islands, cores, or external boundary with angular velocities that are concentrated near the values of the boundary velocities (see Fig. 28) of the corresponding coherent structures (islands, etc.). Dynamics of tracers can be characterized by the time-dependence of the variance of the rotational phase θ

$$\langle(\theta - \langle\theta\rangle)^2\rangle \sim t^{\mu} \tag{365}$$

and Poincaré recurrences distribution density for the tracers, which appears to be of the power law

$$P(\tau) \sim \text{const.}/\tau^{\gamma} \tag{366}$$

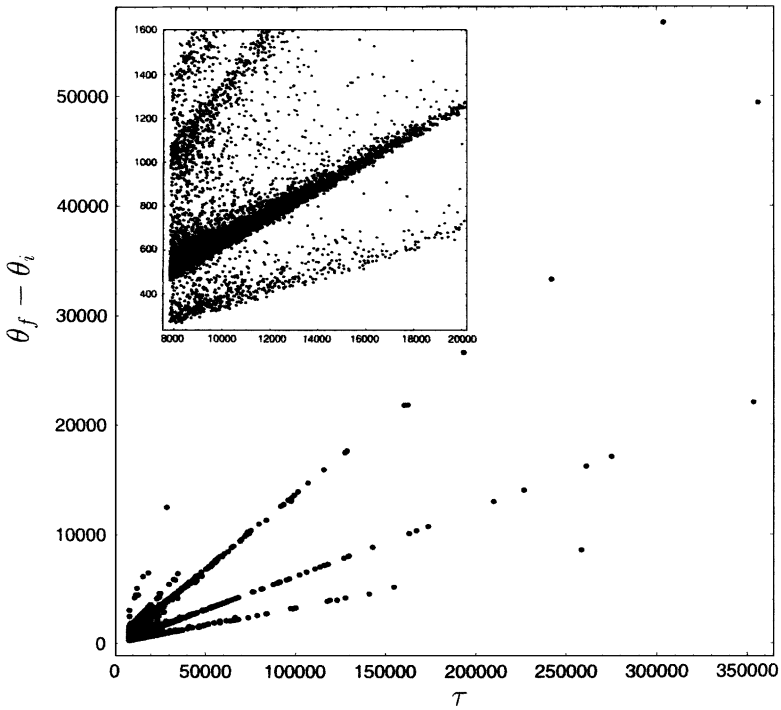

Fig. 28. Scatter plot of flights lengths (net angle covered during the cycle, $\theta_+ - \theta_i$) vs. trapping time. The points are concentrated around the straight lines with the slopes that correspond to the velocities of the rotations around the islands, cores, and external boundary in Fig. 25.

as is presented in Fig. 29. The value of μ oscillates near $\mu = 1.55$ and the tail of $P(\tau)$ has $\gamma = 2.66$. These values are in a good agreement with the law $\gamma = 1 + \mu$. They also demonstrate the closeness of μ to the specific value $\mu_0 = 1.5$ discussed in Section 10.5.

When κ_j are of different signs, the collapse of the vortices is possible for special initial conditions. One can study chaotic advection when the vortex dynamics is close to collapse, i.e. the collapse never appears but the three vortices can approach each other very close. This regime was studied in Leoncini et al. (2001). To study the phenomenon of stickiness, an interval of 10 periods T of the repeating of the three-vortex configuration was selected and the mean velocity V of each tracer was calculated. The distribution function of V (see Fig. 30) shows four sharp maxima around special values of V. Then the Poincaré section of the same tracer was plotted with marks of the tracer's location when the tracer velocity belongs to one of these special values of V or nearby (Fig. 31). All these locations, without exclusions, belong to the vicinities of the vortex cores, or the islands' boundaries, or the external boundary.

The moments

$$M_m \equiv \langle |\theta - \langle\theta\rangle|^{2m} \rangle \sim t^{\mu(m)} \tag{367}$$

have different transport exponents $\mu(m)$ with the anomalous values of $\mu(1)$ depending on how close are the initial conditions to their values of collapse. The closer they are to their values of collapse,

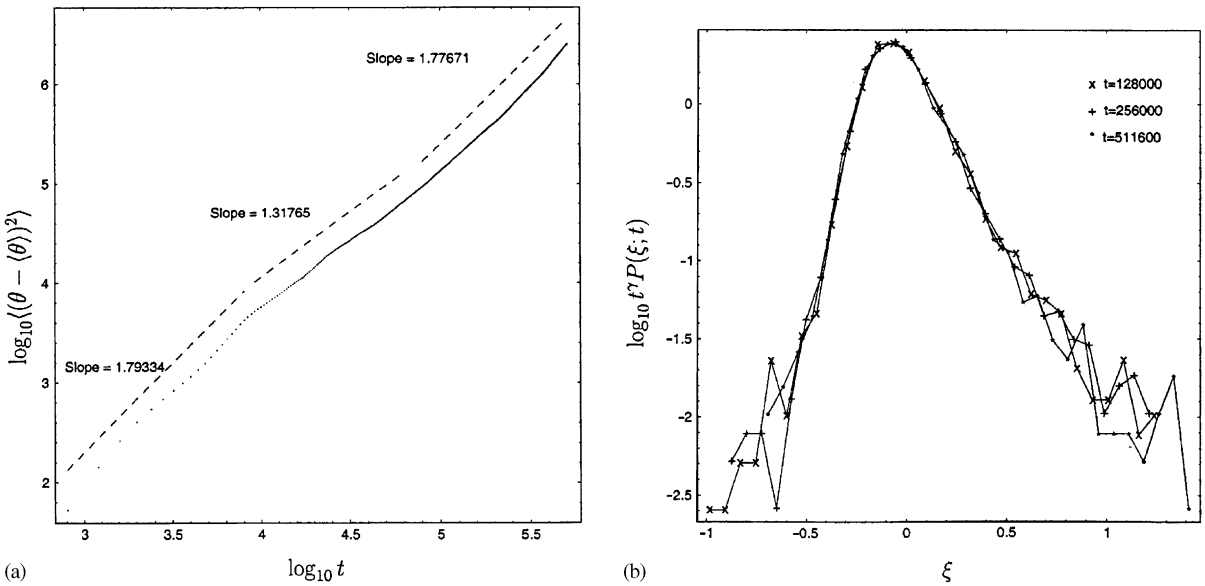

Fig. 29. Anomalous features of tracer dynamics in the 3-vortex flow: (a) the angular variance vs. time; the slopes indicate the transport exponent μ; (b) scaled probability densities of tracer angular displacement ξ for different time instants (Kuznetsov and Zaslavsky, 2000).

the higher is the recurrence exponent γ due to better mixing properties of the tracers. For the values $m > 1$, the $\mu(1) \to 2$ as it should happen when the ballistic parts of trajectories prevail. It was also shown in Leoncini et al. (2001) that the kinetics of tracers can be described by the FKE in the form

$$\frac{\partial^\beta P(\theta,t)}{\partial t^\beta} = \mathscr{D}\,\frac{\partial^\alpha P(\theta,t)}{\partial|\theta|^\alpha} \tag{368}$$

with $\beta \sim 3/2$ and $\alpha = 2 \pm \delta$ with $\delta \sim 0.1$–0.2. The value of $\mu(2)$ was in the interval (1.4–1.6) depending on the control parameter.

12.5. Chaotic advection in many-vortices flow

When number of vortices (see (361)) $N > 3$, the vortex flow can be chaotic by itself and particles advection should be studied in a different way than in the case of $N = 3$. It was shown in Laforgia et al. (2001) that for a system of four same sign κ_j vortices, "collisions" of, mainly, two vortices play the most important role in the anomalous transport of tracers. "Collision" means that two vortices approach each other on the distance of about two core radii. During a collision, tracers can be captured into the cores or escape from the cores. Being captured to a core, the tracers travel together with the corresponding vortex. It was also found that the process of the collision lasts a long time, i.e. two vortices create a pair as a coherent structure. The paired vortices can travel

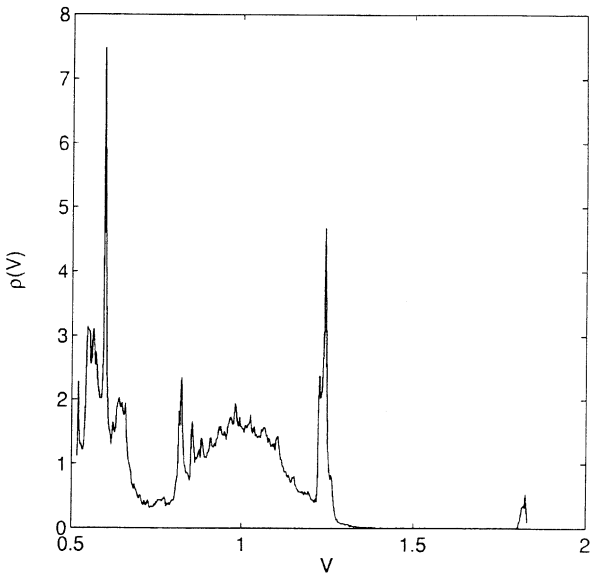

Fig. 30. Distribution $\rho(V)$ of the tracer velocities shows four sharp maxima that corresponds to the tracer stickiness to the cores, islands and the boundary of the domain as it is shown on Fig. 29 (Leoncini et al., 2001). The color variant of Fig. 31 shows a passive particle wandering in the flow generated by 3 point vortices. Positions of the vortices are indicated by two pulses (positive direction) and a circle (negative direction). By different colors we mark different intervals of the particle velocity. The picture shows the zones where the particle stucks, i.e. boundaries of the impermeable islands and the external boundary of the chaotic motion. The data and the Fig. 31 are from Leoncini et al. (2001). The color plate is generated by Misha Zaslavsky on the basis of the data.

almost regularly fairly long time (a flight) and, at the same time, the captured tracers participate in the flight.

In this way the stickiness to the cores proves to be the main mechanism that could define the critical exponents of particles advection. The corresponding FK was obtained in Leoncini and Zaslavsky (2002), where the system of 16 vortices was considered. The results of this work can be summarized in the following way:

$$\mu_v(1) = 1.82, \ \gamma_{\rm tr} = 2.82 \quad \text{(tracers for 4 vortices)} ,$$

$$\mu_v(1) = 1.80, \ \gamma_p = 2.68 \quad \text{(16 vortices)} ,$$

$$\mu_v(1) = 1.77, \ \gamma_{\rm tr} = 2.82 \quad \text{(tracers for 16 vortices)} , \tag{369}$$

where we use the notations: the first and the second line are related to the 4- and 16-vortices systems only (no tracers), and the third line is related to tracers advected by 16-vortices flow. The exponents $\mu(m)$ were calculated for the length s_j of jth vortex trajectory and then averaged over all vortices and about 10^3 trajectories during $t \sim 10^6$:

$$M_m = \langle |s_j - \langle s_j \rangle|^{2m} \rangle \sim t^{\mu_v(m)} . \tag{370}$$

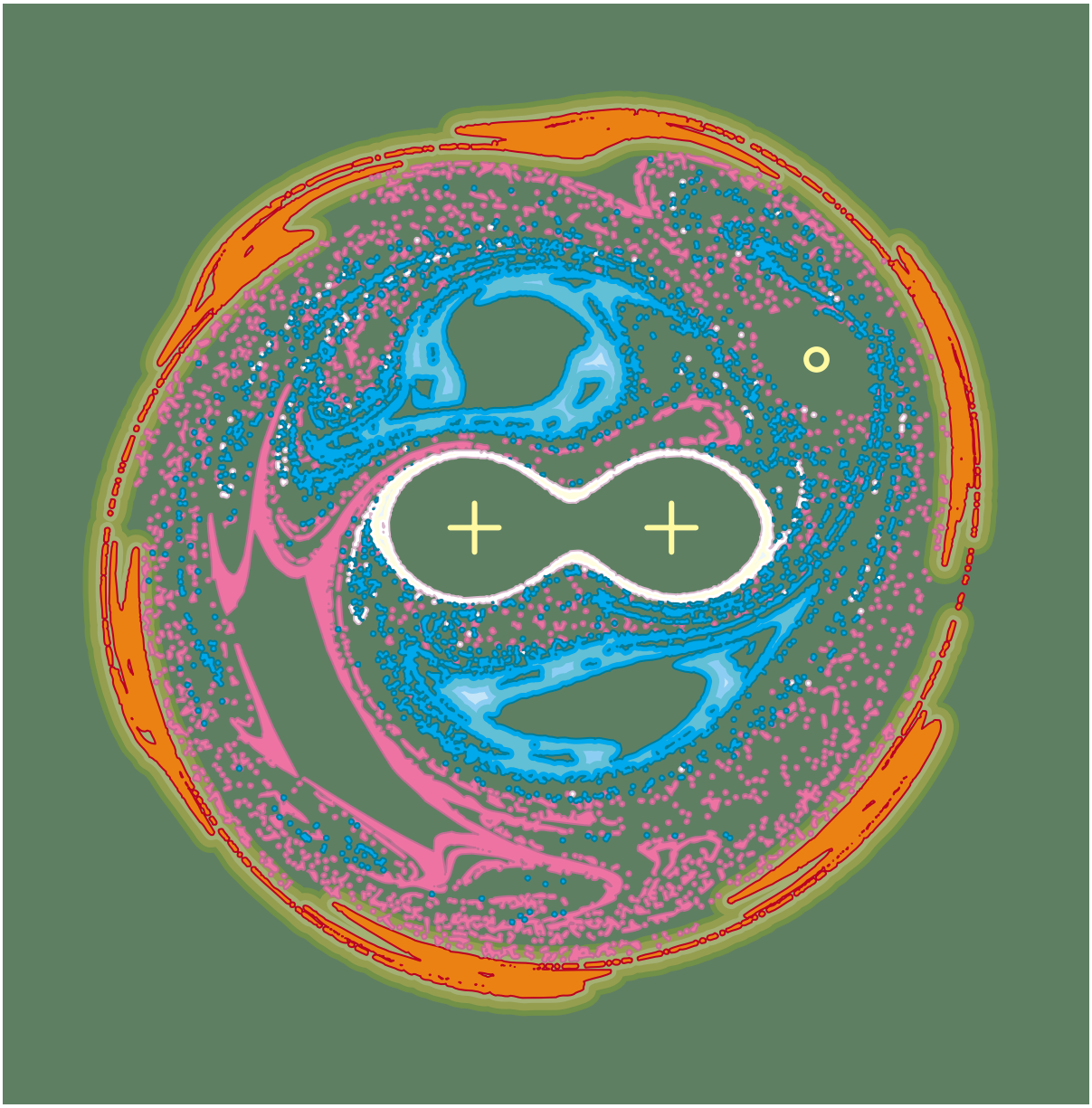

Fig. 31. Poincaré section of a tracer in the 3-vortex demonstrates stickiness to different boundaries (Leoncini et al., 2001); the colour pattern was modified by M. Zaslavskiy.

For $m > 2$, $\mu_v(m) \to 1.9$–2 since only several ballistic pieces of trajectories defines the high moment exponent $\mu_v(m)$.

As it was mentioned above, the pairing of vortices appears for a time distributed as

$$\rho_p(t) \sim \text{const.}/t^{\gamma_p} \ . \tag{371}$$

The first two lines in (369) show that there is no significant difference between 4 and 16 vortices dynamics, and that

$$\gamma_p \approx 1 + \mu_v(1) \ . \tag{372}$$

The last line in (369) shows that advected particles have the same transport exponent $\mu(1)$ as the vortices by itself, and the same is for the pairing exponent γ_p for the vortices and the trapping exponent $\gamma_{\rm tr}$ for the tracers.

12.6. Finite-size Lyapunov exponents and stochastic jets

In this section we emphasize two additional tools important for the study of the FK. The standard definition of the Lyapunov exponent

$$\sigma = \lim_{t\to\infty} \lim_{\epsilon\to 0} \ln \frac{\ell(t)}{\epsilon} \tag{373}$$

with $\epsilon = \ell(0) \ll \ell(t)$ is difficult to utilize in complicated models since the time of simulation and the size of systems are always finite, i.e. the limits in (373) cannot be actually performed. This difficulty sometimes is negligible if the system under consideration possesses good mixing properties, but it is definitely not the case in systems with flights, trappings and other types of anomalies that occur as non-Gaussian fluctuations.

To avoid the described difficulty, one can consider *finite size Lyapunov exponents*

$$\bar{\sigma}_T = \frac{1}{T} \left\langle \ln \frac{\ell(T)}{\epsilon} \right\rangle \ ,$$

$$\bar{\sigma}_L = \frac{1}{t(L)} \left\langle \ln \frac{L}{\epsilon} \right\rangle \tag{374}$$

with fixed T or fixed L and an appropriate small ϵ. The averaging can be performed either over a long trajectory or over many trajectories, and $t(L)$ is a time during which two initially ϵ-separated trajectories diverge to the fixed value L (see for example in Bohr et al., 1998).

In Leoncini and Zaslavsky (2002), the magnitude was considered:

$$\sigma_L = \frac{1}{t(L)} \ln \frac{L}{\epsilon} \tag{375}$$

with $L \gg \epsilon$ but very small compared to the size of the system, and without averaging. In this way, many different values of σ_L can be collected and their distribution $\rho(\sigma_L)$ can be considered. For advection in the system of 16 vortices, $\rho(\sigma_L)$ is shown in Fig. 32. The distribution has a peak for small values of $\sigma_L \to 0$, which corresponds to the stickiness of trajectories. One can expect that all trajectories started at the domains of small σ_L should travel almost together for a fairly long distance with a negligible dispersion. Such bunches of trajectories were called *stochastic jets* (Afanasiev et al., 1991; Leoncini and Zaslavsky, 2002). In the latter paper the stochastic jets were demonstrated as a kind of coherent structures. They were found in the stochastic sea by a special type of the simulation detection. Stochastic jets is a typical phenomenon for the FK.

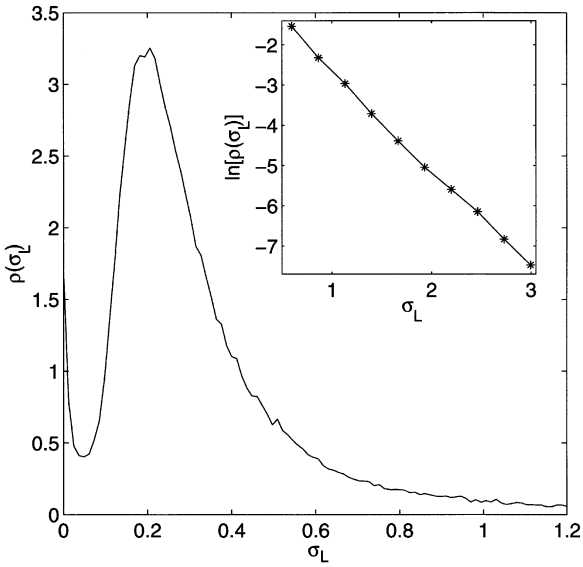

Fig. 32. Distribution of the finite size Lyapunov exponent for tracers in the multi-vortex flow. Insertions shows exponential decay of the distribution for large values of σ_L (Leoncini and Zaslavsky, 2002).

13. Fractional kinetics in plasmas

Confinement plasma is a system for which the problem of anomalous transport and chaos has been discussed for a number of years. Unfortunately, this system is very complicated and the transport is not so transparent as in the chaotic advection. This makes the study of FK in plamsas more fragmented, partly related to some elementary processes and partly related to simplified models. In this section we discuss only those cases of transport in plasmas that use the FK in an explicit way.

13.1. Fractional kinetics of charged particles in magnetic field

One-particle dynamics in different configurations of magnetic field can be considered as the simplest object to study anomalous transport and FK in plasma. Nevertheless, the complicated structure of the realistic magnetic field and the presence of other fields, that act across the magnetic field, make the particle transport problem fairly complicated. The phenomenon of "stochastic streaming" (flights) was observed in numerous simulations. A model of 2 1/2 degrees of freedom in Afanasiev et al. (1991):

$$H = \frac{1}{2m_0}(\dot{x}^2 + \dot{z}^2) + \frac{1}{2} m_0 \omega_0^2 x^2 - \mathrm{e}\phi_0 T \cos(xk_x + zk_z) \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{376}$$

describes a particle motion in magnetic field B_0 and in the wave packet with a potential amplitude ϕ_0; $\omega_0 = \mathrm{e}B_0/m_0c$ is cyclotron frequency, and $\mathbf{k} = (k_x, k_z)$ is the wave number. System (376) can be

reduced to a map:

$$u_{n+1} = [u_n + K\sin(v_n - Z_n)]\cos\bar{\alpha} + v_n\cos\bar{\alpha}\ ,$$

$$v_{n+1} = -[u_n + K\sin(v_n - Z_n)]\sin\bar{\alpha} + v_n\cos\bar{\alpha}\ ,$$

$$w_{n+1} = w_n + K\bar{\beta}^2\sin(v_n - Z_n)\ ,$$

$$x_{n+1} = Z_n + \bar{\alpha}w_{n+1} \tag{377}$$

with the notations:

$$u = k_x\dot{x}/\omega_0, \quad v = -k_x x, \quad w = k_z\dot{z}/\omega_0, \quad Z = k_z z\ ,$$

$$\bar{\beta} = k_z/k_x, \quad \bar{\alpha} = \omega_0 T\ . \tag{378}$$

Eqs. (377) describe the coupled standard map and web map. The analysis of the paper (Afanasiev et al., 1991) showed different types of anomalous transport for the moments:

$$\langle R^2\rangle = \langle u^2 + v^2\rangle \sim \text{const.}\cdot t^{\mu_\perp} \tag{379}$$

with values of $\mu_\perp$ superdiffusive ($\mu_\perp > 1$) and subdiffusive ($\mu_\perp < 1$). One can apply the CTRW analysis with fractional exponents (α, β) to explain the results. It was shown in Petrovichev et al. (1990) and Weitzner and Zaslavsky (2001) that not only the diffusion constant can reveal the anisotropy of transport, but also different values of the transport exponent can indicate a strong anisotropy of the transport processes, i.e. one can expect that $\mu_\parallel$ in

$$\langle Z^2\rangle \sim t^{\mu_\parallel} \tag{380}$$

differs from $\mu_\perp$. Another more detailed analysis of transport for the model (376) can be found in Zaslavsky et al. (1989a, b); Neishtadt et al. (1991).

In the case of $k_z = 0$, i.e. $\bar{\beta} = 0$, $Z_0 = \omega_0 = 0$, dynamics of particles is in the (u, v)-plane only and the system is reduced to the web map (Zaslavsky et al., 1991):

$$u_{n+1} = (u_n + K\sin v_n)\cos\bar{\alpha} + v_n\sin\bar{\alpha}\ ,$$

$$v_{n+1} = -(u_n + K\sin v_n)\sin\bar{\alpha} + v_n\cos\bar{\alpha} \tag{381}$$

which for the case $\bar{\alpha} = 2\pi/4$ coincides with (11). Anomalous transport and FK for (11) was already described (see also Niyazov and Zaslavsky, 1997; Leboeuf, 1998). Ballistic orbits in the web map were found in Iomin et al. (1998) and a Hamiltonian theory of the ballistic islands was given in Rom-Kedar and Zaslavsky (1999).

Similar to the web map, anomalous transport has been observed for the separatrix map (see Section 9.2 based on a bifurcation that open the ballistic islands and creates the stickiness to it (Zaslavsky and Abdullaev, 1995; Abdullaev and Zaslavsky, 1996; Abdullaev and Spatchek, 1999; Abdullaev, 2000; Abdullaev et al., 2001)) and, as a result, superdiffusion. For the Poincaré recurrences distribution, it was found $\gamma_{\text{rec}} \approx 2.5$ (Abdullaev, 2000). These results indicate the characteristic features of the FK of transport along the stochastic layer or stochastic web although the investigation that leads to the FKE is not done yet. Another set of publications consider charged particle dynamics in a constant magnetic field and random electric field using a corresponding FKE (Chechkin and Gonchar, 2000; Chechkin et al., 2002).

13.2. Magnetic fieldlines turbulence

Magnetic fieldline turbulence is an analogy to the Lagrangian, or streamlines, turbulence. The analogy follows not only from the similarity of Eqs. (347) and (351). A charged particle in a strong magnetic field spirals along the fieldline with a small Larmour radius. The particle trajectory follows exactly the fieldline after neglecting the Larmour rotations. One can say in this case that the charged particle is advected by the magnetic field.

A typical simplified problem that appears for particles near the plasma edge can be formulated as

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}z} = \mathbf{B}_0(\mathbf{r}) + \delta\mathbf{B}(\mathbf{r}, z) \ ,$$
$$\delta\mathbf{B}(\mathbf{r}, z) = \sum_{\mathbf{k}} \delta\mathbf{B}_{\mathbf{k}} \exp \mathrm{i}(\mathbf{k}\mathbf{r} + \kappa z) \ , \tag{382}$$

where $\mathbf{r} = (x, y)$, z is the coordinate along the toroidal axis and it plays a role of the time variable, $\delta\mathbf{B}$ is a field of perturbation. Typically both fields $\mathbf{B}_0$ and $\delta\mathbf{B}$ are known, and problem (382) is reduced to one similar to (355). Anomalous transport in the form

$$\langle(\Delta x)^2\rangle \sim t^{\mu_x}, \quad \langle(\Delta y)^2\rangle \sim t^{\mu_y} \tag{383}$$

has been observed in simulation by Zimbardo et al. (2000a, b) with the values of μ_x, μ_y between 1.3 and 4.1. Some kind of interpretation of these results is based on the Lévy walk model.

Similar to (382) problem appears for the particle dynamics in the so-called ergodic divertor. The results of the fractional analysis of the problem are in Abdullaev et al. (1998, 2001), Balescu (1997), Vlad et al. (1998) and Lesnik and Spatschek (2001).

13.3. Test particles in self-consistent turbulent plasma

The previous sections were related to strongly simplified models with separated particles and fields. In fact, any reasonable model of plasma in toroidal devices consists of a set of equations with coupled fields and particles. A simplification can be done on the level of such models which, typically, can be reduced to some fluid dynamics type equations. This part of the investigation of different physical processes is similar to the study of fluid turbulence in complicated conditions and many-parametric systems. Let $\boldsymbol{\Psi} \equiv \{\phi, n, \ldots\}$ is a set of the functions that describe the corresponding media. Then the corresponding equations have a form

$$\frac{\partial\boldsymbol{\Psi}}{\partial t} = \hat{\mathscr{L}}\boldsymbol{\Psi} + (\boldsymbol{\Psi}\hat{\mathscr{N}})\boldsymbol{\Psi} \ , \tag{384}$$

where $\boldsymbol{\Psi} = \boldsymbol{\Psi}(\mathbf{r}, t)$, $\hat{\mathscr{L}}$ and $\hat{\mathscr{N}}$ are linear and non-linear operators that include different space–time derivatives. As an example, let us mention the so-called Hasegawa–Mima equation

$$\frac{\partial}{\partial t}(\phi - \nabla^2\phi) = \gamma\{\phi, \nabla^2\phi\} - \beta_r \frac{\partial\phi}{\partial x} = 0 \ , \tag{385}$$

where ϕ is electrostatic potential, braces $\{A, B\}$ mean the Jacobian of A and B, γ and β_r are some functions of coordinates. Under fairly general conditions, the solution for Eq. (385) corresponds to a turbulent motion with a presence of coherent structures.

It is always a difficult problem to visualize the solution and to provide its physical interpretation. Simply speaking, one cannot see a flow. Instead, we can put tracers or passive particles and observe their kinetics. This way of consideration consists of two problems: dynamics (kinetics) of the media and kinetics of tracers in the regular or turbulent media as well as connection between the two types of description of the media properties. The best example of these two approaches is the Kolmogorov spectrum for a turbulent flow and the Richardson law for particles dispersion in the same turbulent flow.

Investigation of these two approaches in plasma physics is very important for the fusion program but, also, hard to be worked out. Different results on this way can be found in Benkadda et al. (1997a, b), Naulin et al. (1999), Annibaldi et al. (2000), Zaslavsky et al. (2000), Carreras et al. (2001), Beyer and Benkadda (2001), Ragot and Kirk (1997), Milovanov and Zelenyi (2001) and Zaslavsky et al. (2001). They show an evidence of FK for some fairly general conditions. It could be reasonable to assume that numerous coherent structures, presented in turbulent flow, impose non-Gaussian types of kinetics.

14. Fractional kinetics in potentials with symmetry

14.1. Potentials with symmetry

Restricting ourselves by 2-dimensional case, one can write a Hamiltonian of particle dynamics in a potential $V(x, y)$ as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) \ . \tag{386}$$

We also assume that $V(x, y)$ has a symmetry of an order q, i.e.

$$V_q(\mathbf{r}) = V \sum_{j=1}^{q} \cos(\mathbf{r}\mathbf{e}_j) \tag{387}$$

with $\mathbf{r} = (x, y)$ and $\mathbf{e}_j$ as the unit vectors that form a regular star:

$$\mathbf{e}_j = (\cos(2\pi j/q), \sin(2\pi j/q)), \quad j = 0, 1, \ldots, q-1 \ . \tag{388}$$

The potential V_q has the crystal symmetry for $q = 1, 2, 3, 4, 6$ or quasi-crystal symmetry for all other q (see in Zaslavsky et al., 1991). Particle dynamics is integrable for $q = 1, 2, 4$ and non-integrable (in general) for all other q. In the non-integrable cases, there are stochastic webs that tile the plane (x, y) with an approximate symmetry of the order q. Chaotic dynamics along the webs can be considered as a nice example of FK, at least due to the presence of ballistic modes with q-symmetry (for $q = 4$ the ballistic modes were found in Iomin et al., 1998).

Another example of the periodic potential is the so-called Lorentz gas models, i.e. an infinite periodic in one or two directions a set of scatterers that creates chaotic dynamics of point particles performed the billiard type motion (see Section 11.4 where Sinai-billiard and Cassini-billiard were mentioned). In this section we consider only dynamics of the (387) type and postpone until the Section 15 a discussion of different types of infinite billiards.

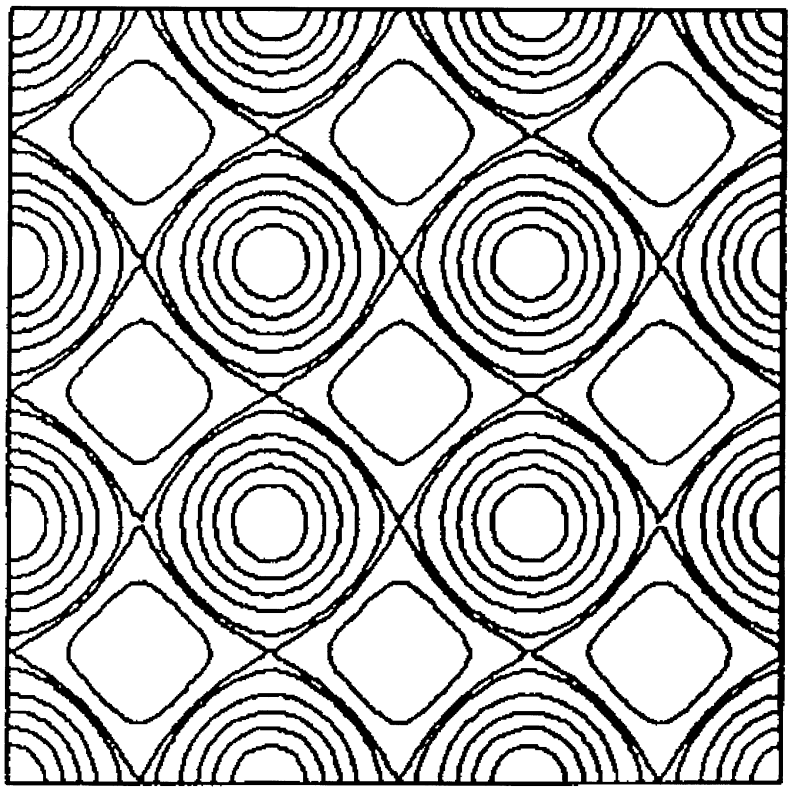

Fig. 33. Equipotential contours for the egg-crate potential and $A = B$.

14.2. Egg-crate potential

The case $q = 4$ is integrable and chaotic dynamics appears only due to a perturbation. Consider the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + A\cos x + B\cos y + \epsilon\cos x\cos y \tag{389}$$

with a perturbation parameter ϵ. The surface $V = V(x, y)$ has a shape of the egg-crate for $A = B$. The last term in (389) breaks the 4-fold symmetry of the potential and makes the dynamics chaotic is some domains of phase space. Kinetics in model (389) has been studied with respect to different applications in statistical physics, electron dynamics in lattice, plasma physics, advection in fluids and others (Machta and Zwanzig, 1983; Bagchi et al., 1985; Geisel et al., 1987a, b; Kleva and Drake, 1984; Horton, 1981; Sholl and Skodje, 1994; Klafter and Zumofen, 1994; Cherinikov et al., 1987 and Chaikovsky and Zaslavsky, 1991).

The theory of chaotic dynamics, stochastic webs and their width in model (389) were considered in Chaikovsky and Zaslavsky (1991). Unperturbed ($\epsilon = 0$) equipotential contours are in Fig. 33. There are different "channels" with chaotic dynamics inside of them, and each channel corresponds to a stochastic layer (Fig. 34). At some critical value of the energy $E_c = E(\epsilon)$ of a particle different

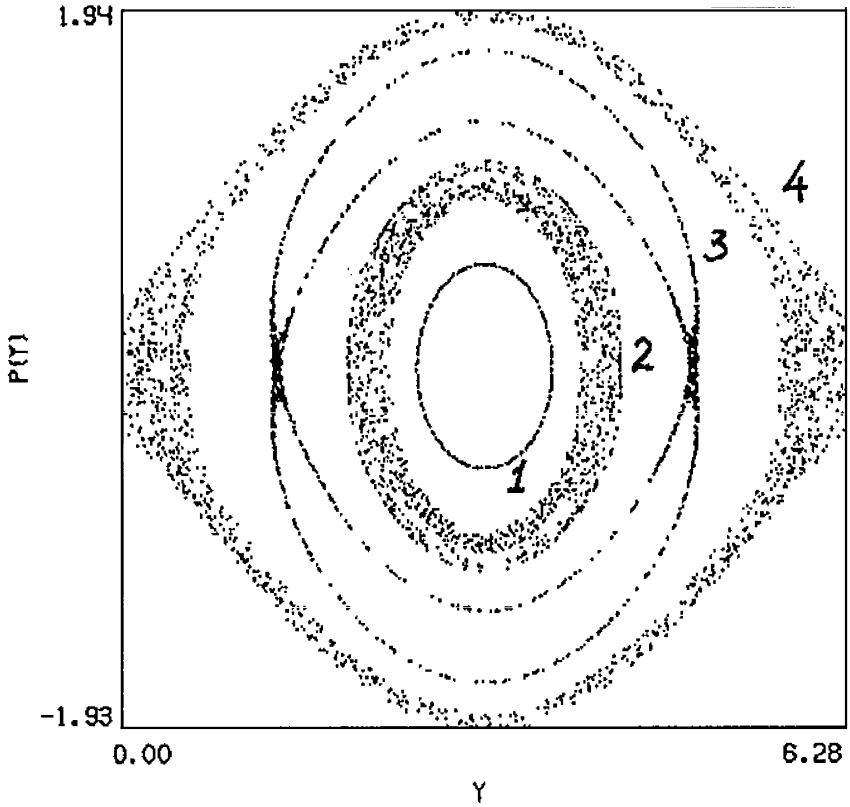

Fig. 34. Separated stochastic layers 2 and 4 correspond to different channels of chaotic dynamics.

stochastic layers merge (Fig. 35) and the percolation appears (Fig. 36). The process of particle wandering is unbounded in two directions with the mean displacement

$$\langle R\rangle = \langle (x^2 + y^2)^{1/2}\rangle \sim t^{\mu/2} \tag{390}$$

and $\mu = \mu(\epsilon) > 1$. A sample of trajectory in Fig. 37 shows extremely long flights which can be strongly anisotropic depending on the value of energy E.

Let us mention that at the FK relation to the problem of percolation was broadly discussed in the review of Isichenko (1992) and more recently in Milovanov (2001).

14.3. Fractional kinetics in other potentials

As it was mentioned in Section 14.1, dynamics in V_q for $q=3,5,6,\ldots$ is chaotic and corresponding anomalous transport properties were investigated in a number of works (Zaslavsky et al. (1989a, b) for $q=3$ and 5; Zaslavsky et al. (1994) for $q=3$ and tracers in fluids; Panoiu (2000) for $q=3$). The potential for $q=3$ is

$$V_3 = \cos x + \cos\left[\frac{1}{2}(x+\sqrt{3})\right] + \cos\left[\frac{1}{2}(x-\sqrt{3})\right]. \tag{391}$$

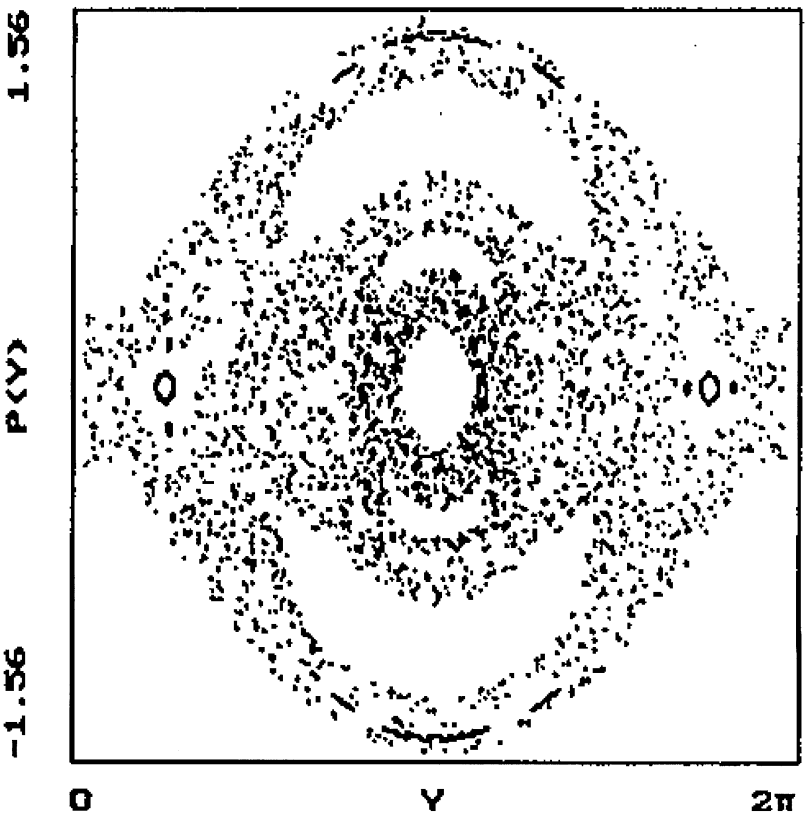

Fig. 35. Merge of the stochastic layers from the previous figure corresponds to a kind of percolation process when particles can wander in all directions (see the next figure).

An example of a sticky island and flights is shown in Fig. 38 Panoiu (2000). More results on the Poincaré recurrences, islands topology, and transport exponents for V_3 can be found in Panoiu (2000).

For quasicrystal type symmetry, the diffusion can be localized in some part of the coordinate space since some potential barriers can stop a percolation for not too high energy (Zaslavsky et al., 1989a, b). That means that the anomalous behavior of particles can appear in some domains of fairly large size domains without percolation between the domains.

More complicated dynamics for a particle in a periodic potential and a constant magnetic field B along axis z was considered in Rakhlin (2000). Hamiltonian of the problem is

$$H(x, y; p_x, p_y) = \frac{1}{2m}[(p_x + eB_y/2)^2 + (p_y - eB_x/2)^2] + V_0[2 + \cos(2\pi x/a) + \cos(2\pi y/a)] \ . \tag{392}$$

The phase space topology is very rich. It has, depending on the control parameters and energy, HIT, stochastic layer trap, stochastic net-trap (see Sections 9.3–9.5). Due to that, different kind of the anomalous transport were identified and demonstrated by a simulation.

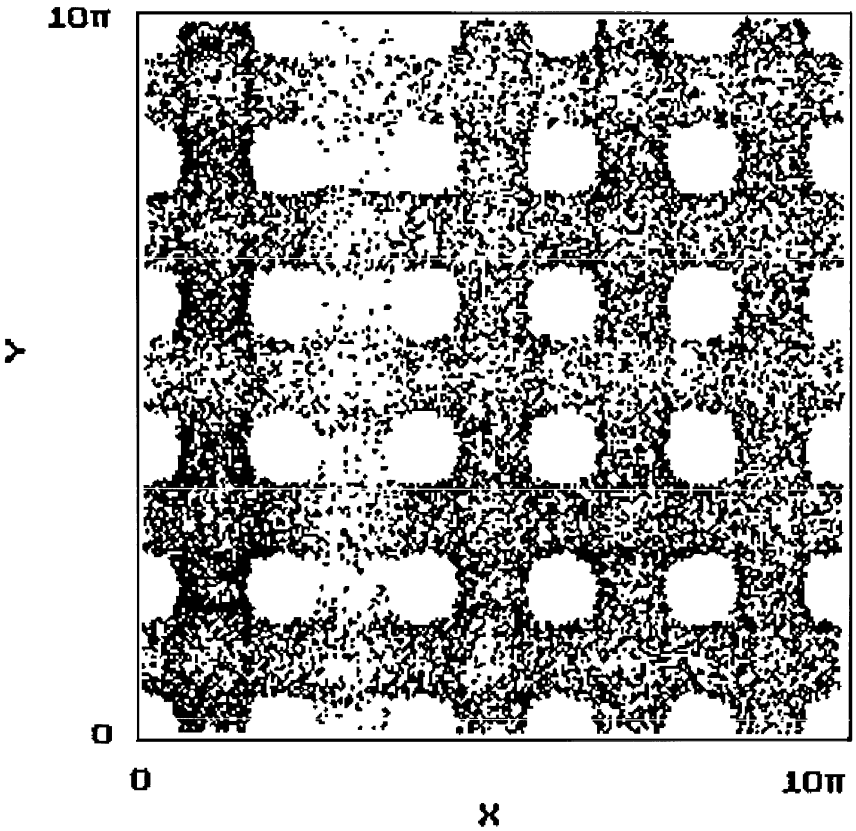

Fig. 36. Percolation in the configuration space. The points represent the corresponding Poincaré section.

The lattice potential $V_q(x, y)$ can be created by the optical laser beams. It was shown the existence of Lévy walks in such potential (Marksteiner et al., 1996). For multi-particle systems see in Latora et al. (2000).

14.4. Anomalous transport in a round-off model

Computerized consideration of dynamical models leads to a number of questions on what is lost and what is gained in an approximate scheme compared to the real results. From some general point one can state that discretization of differential equations and their replacement by the difference ones, leads to occurrence of chaotic trajectories instead of some regular ones (Zaslavsky, 1981). A new and interesting approach to the issue occurs in Vivaldi (1994) where the transport of the round-off error near the periodic orbits was considered from the dynamics of some Hamiltonian model.

In a set of publications (Lowenstein et al., 1997; Lowenstein and Vivaldi, 1998; Kouptsov et al., 2002), the problem of propagation of the round-off was formulated in the following way. The

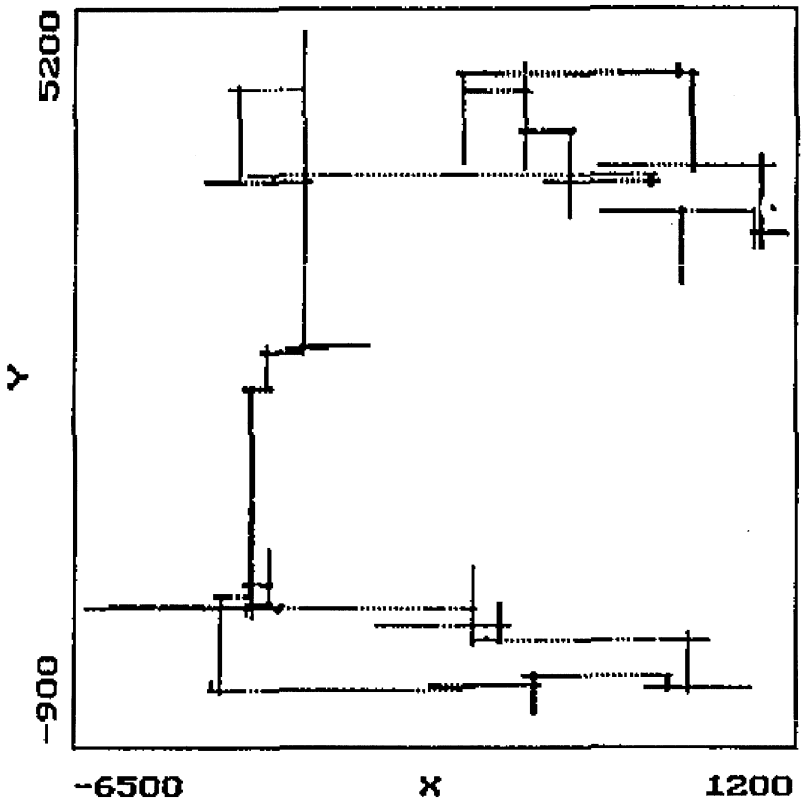

Fig. 37. Random walk after the percolation shows very long flights.

original problem is a map:

$$\Psi_1 \mathbf{R}^2 \to \mathbf{R}^2, \quad \Psi(x,y) = (\lambda x - y, x) \tag{393}$$

with $\lambda = e\cos(2\pi v)$ and a rational rotational number $v = p/q$, i.e. (393) describes the q-periodic dynamics. The discretization was considered as displacement of the phase space by discretized phase space, i.e. $\mathbf{R}^2 \to \mathbf{Z}^2$ and the map

$$\phi : \mathbf{Z}^2 \to \mathbf{Z}^2, \quad \phi(x,y) = (\lfloor \lambda x \rfloor - y, x) \ , \tag{394}$$

where $\lfloor \cdot \rfloor$ is the floor function, and the same λ as in (393). A sample of phase portrait for $q = 5$ and ϕ^5 is shown in Fig. 39 (Lowenstein and Vivaldi, 1998). Periodic orbits of ϕ^5, starting at different initial points (x_0, y_0) have different deviations from the original periodic orbit and, moreover, these deviations can be considered as a random variable χ. For the mean value $\langle \chi \rangle$ an exact formula was obtained:

$$\langle \chi \rangle = t^{\beta} f(t) \tag{395}$$

with an exactly calculated irrational value of β, bounded function $f(t)$, and t as a number of iterations of the round-off orbit to return to the starting point.

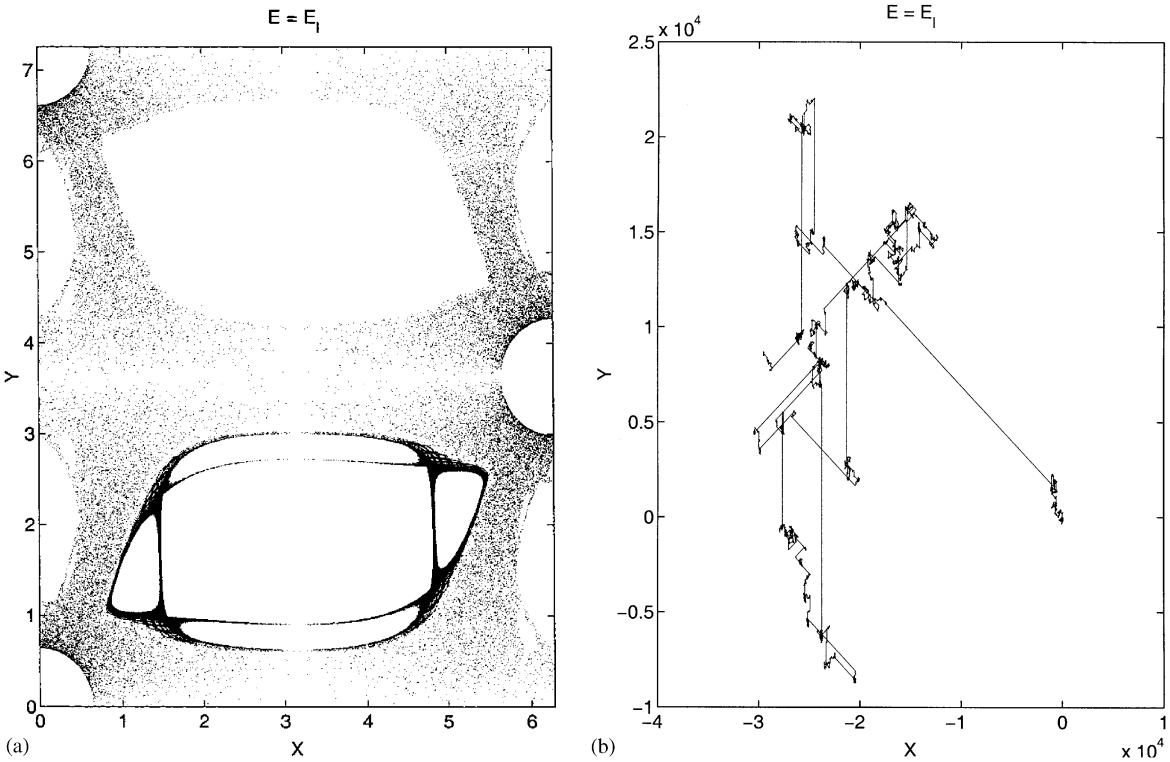

Fig. 38. Stickiness to an island boundary (a) and the flights of the order of more than 10^4 (b) for a particle dynamics in the hexagonal symmetry potential (Panoiu, 2000).

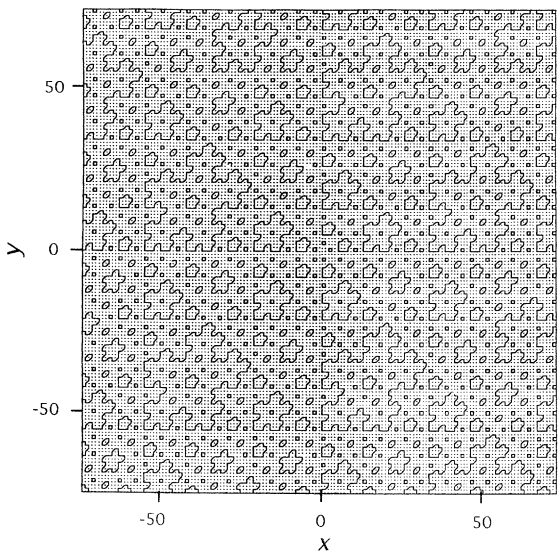

Fig. 39. Closed contours of the "round-off dynamics" (Lowenstein and Vivaldi, 1998).

Similar to the round-off map are maps that describe chaos in digital filters (see Ashwin et al., 1997; Chua and Lin, 1998).

15. Fractional kinetics and pseudochaos

15.1. Pseudochaos

We refer to chaotic dynamics, or simply to chaos, for the case when trajectories are sensitive to small perturbation of initial conditions. Sensitivity means an exponential growth with time of the distance between unperturbed and perturbed trajectories. Typically it is expressed by the property that Lyapunov exponent is finite, i.e. $\sigma > 0$ (see definition of σ in (276)). If the dynamics is not ergodic, as it is for typical Hamiltonian systems due to the islands, even for $\sigma > 0$ the kinetics is non-Gaussian and FK provides a more adequate description of transport properties.

There are systems with $\sigma = 0$ but with, nevertheless, a kind of randomness or unpredictability. Some examples of such systems of the billiard type are in Fig. 40. The study of trajectories in elastic billiards has a relation to the problems of advection and magnetic surfaces structure (Zaslavsky and Edelman, 2001). Another application of billiards with $\sigma = 0$ is in acoustics of bodies with the fractal shape of their boundaries (Sapoval et al., 1991). Trajectories dispersion in these types of billiards is not faster than of the algebraic dependence on time. This kind of randomness will be called *pseudochaos*, or in other words, *algebraic sensitivity*.

Adjoint problem to the dynamics in the billiards of Fig. 40 is the dynamics in the periodically or double periodically continued systems as in Fig. 41 i.e. with infinite covering space. These kinds

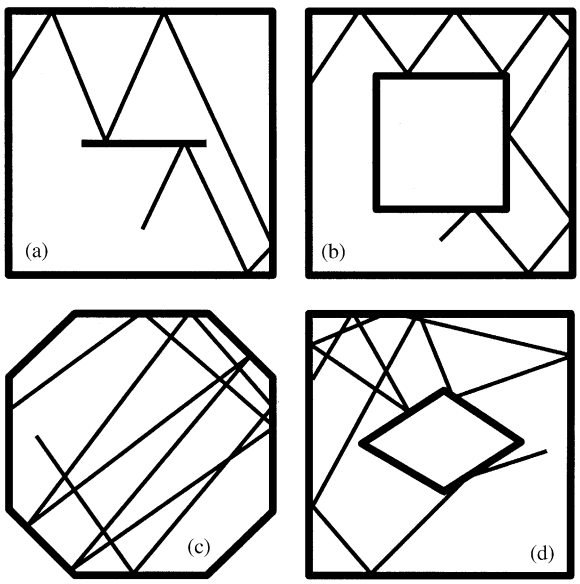

Fig. 40. Examples of billiards with zero Lyapunov exponent.

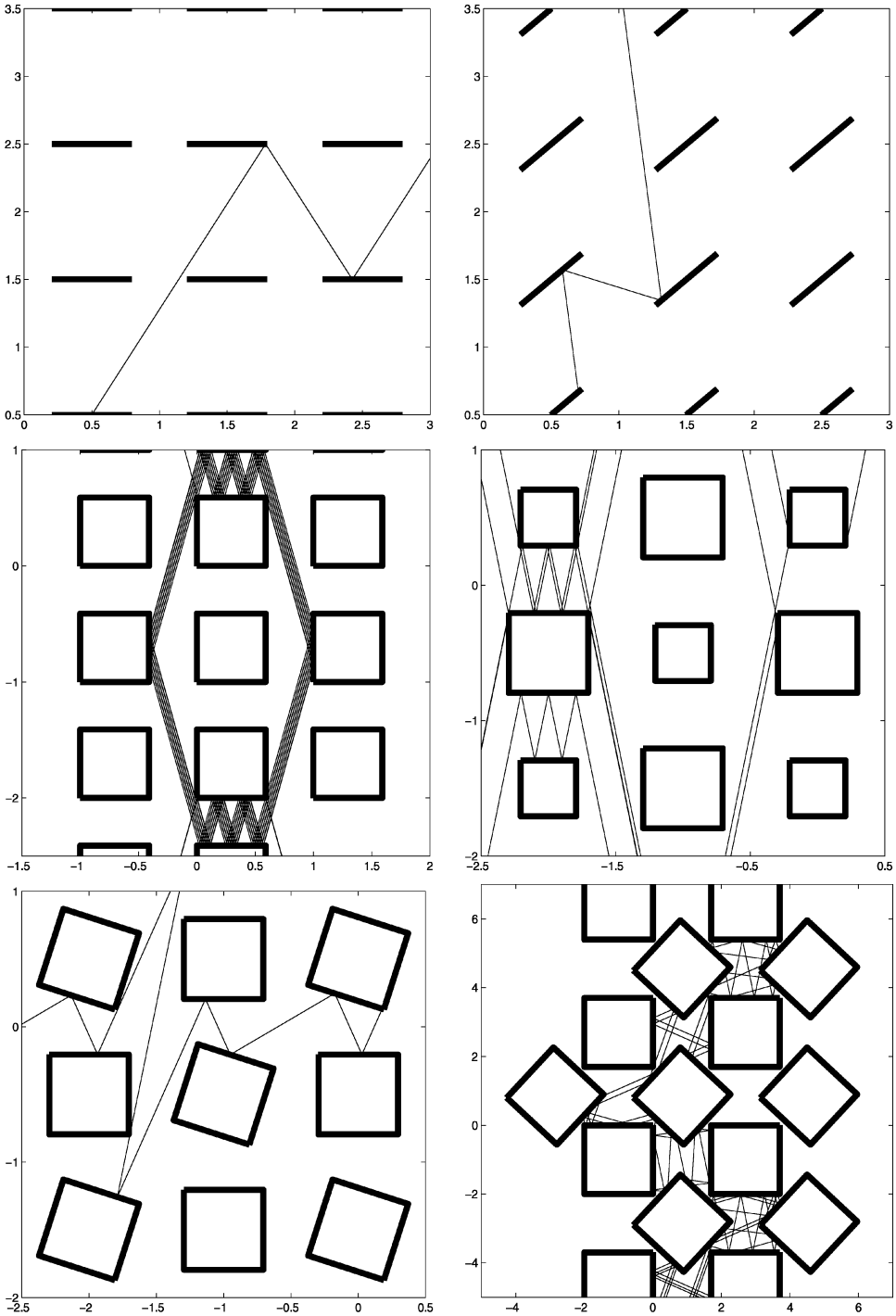

Fig. 41. Periodically continued billiards (billiards in the infinite cover space) form a generalized Lorentz gas (GLG).

of systems we call generalized Lorentz gas (GLZ) and it can be used to study kinetics and transport of particles.

In spite of numerous literature on billiards properties (Galperin and Zemlyakov, 1990; Katok, 1987; Bunimovich and Sinai, 1981; Gutkin, 1986, 1996; Chernov and Young, 2000), rigorous results on the kinetics do not exist yet (see Zorich (1997) for the restricted problem), although there are numerous observations of the anomalous kinetics of trajectories (Artuso et al., 1997, 2000; Casati and Prosen, 1999; Hannay and McCraw, 1990; Lepri et al., 2000; Wiersig, 2000; Zwanzig, 1983). Using the method of FK, some results for the pseudochaotic billiards can be obtained in an explicit form (Zaslavsky and Edelman, 2001; Zaslavsky and Edelman, 2002).

15.2. Fractional kinetics and continued fractions in billiards

It may be the best way to understand characteristic features of the pseudochaotic kinetics to consider a simplified version of billiards, shown in Figs. 40, part (a) and 42, top left: a square with a slit and the equivalent GLG. A trajectory is called rational if the tangent of its angle with a side $\operatorname{tg}\vartheta$ is rational, and irrational in the opposite case. For $\operatorname{tg}\vartheta = p/q$ with mutually prime p and q trajectories are periodic. A trajectory of the billiard can be put on a "pretzel" with two holes (torus with one handle). A sample of an irrational trajectory in GLG is shown in Fig. 42. It demonstrates quasi-periodicity and a kind of self-similarity since the quasi-periods are different for different time scales. The origin of the quasi-periodicity is in a possibility to approximate the fractional part ξ of an irrational $\operatorname{tg}\vartheta$ as a continued fraction:

$$\xi \equiv [a_1, a_2, \dots] = 1/(a_1 + 1/(a_2 + \cdots)) \tag{396}$$

which can be approximated by the approximants of different orders

$$\xi_n \equiv [a_1, \dots, a_n] = p_n/q_n \tag{397}$$

and mutually prime p_n, q_n. In fact, what is seen in Fig. 42 are approximants, i.e. quasi-periodic orbits, of different orders.

It is known that expansion (396) has remarkable scaling properties (Khinchin, 1964): scaling of q_n

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln q_n\right) = \pi^2/12\ln 2 = 1.186\dots \tag{398}$$

and scaling of a_n

$$\lim_{n\to\infty}(a_1\dots a_n)^{1/n} = \prod_{k=1}^{\infty}\left(1 + \frac{1}{k^2+2k}\right)^{\ln k/\ln 2} = 2.63\dots\ . \tag{399}$$

Since the periods of the corresponding approximants $T_n = \text{const.}\ q_n$, we obtain the scaling of the periods

$$\ln\lambda_T = 1.186\dots\ . \tag{400}$$

It was shown in Zaslavsky and Edelman (2002) that the scaling of flights y_n along y corresponding to the periods T_n is

$$\ln\lambda_y \approx \ln[(\lambda_T + \lambda_a)/2] \approx 1.08\ , \tag{401}$$

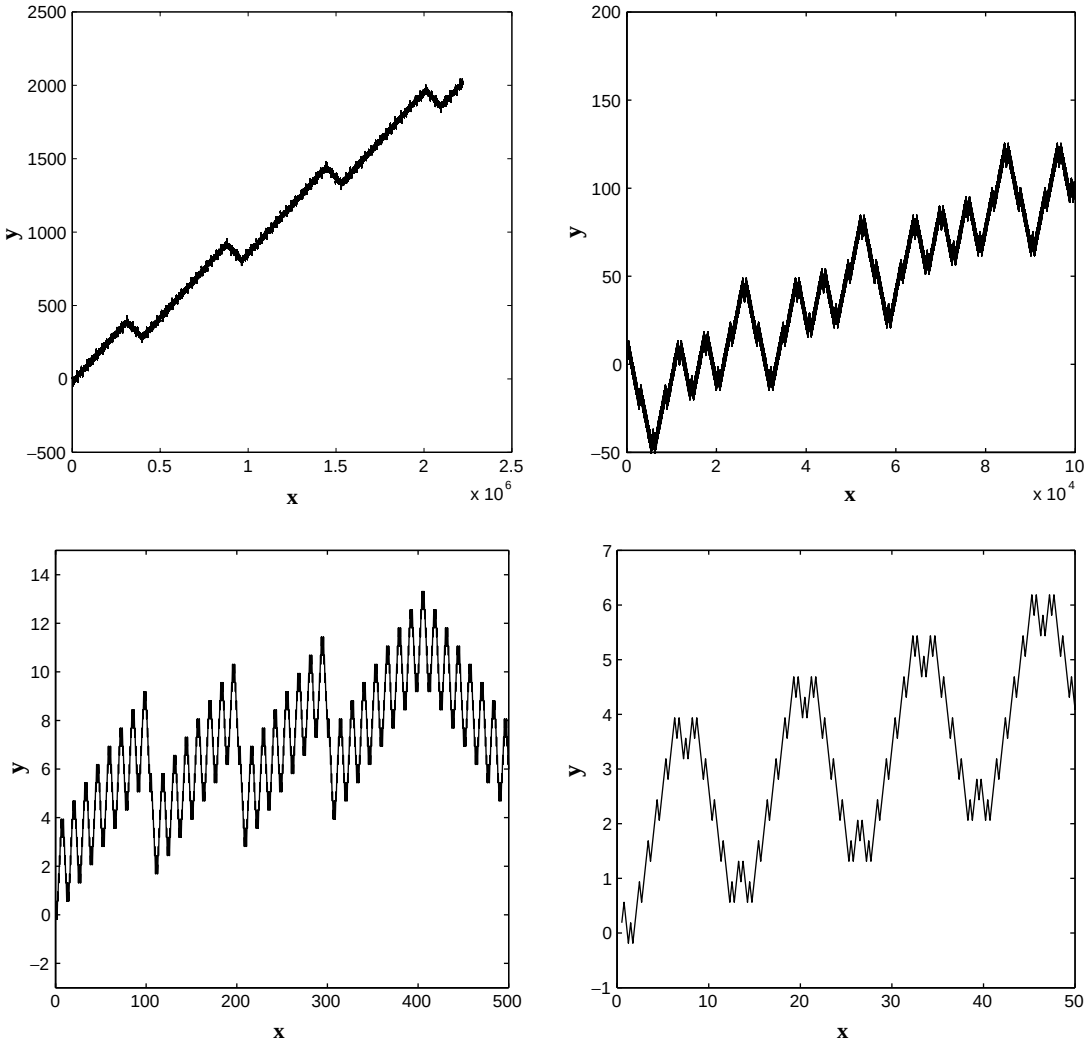

Fig. 42. Samples of a trajectory with different space (time) scales for the square-with-slit billiard.

where $\lambda_a = 2.63$ due to (399). Using the values of λ_y and λ_T and applying (247) with replacement $\lambda_\ell \to \lambda_y$, it was obtained Zaslavsky and Edelman (2002):

$$\mu = 2 \ln \lambda_y / \ln \lambda_T = 1.08 \tag{402}$$

and for log-periodic oscillations (see (253)):

$$T_{\log} = \ln \lambda_T = 1.186 \ . \tag{403}$$

Simulations were performed in Zaslavsky and Edelman (2001) using the ensemble averaging. The ensemble consists of a large number of trajectories launched from a point with different angles. This way of averaging proves to be stable and confirms the results (401) and (402) with a satisfactory

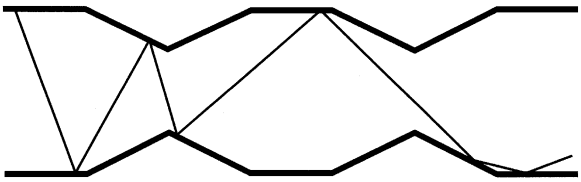

Fig. 43. Wave-guide type of the billiard.

accuracy. It also confirms a possibility to apply the FK equation to trajectories of GLG with the slits as scatterers, i.e.

$$\frac{\partial^\beta F(y,t)}{\partial t^\beta} = \mathscr{D}\,\frac{\partial^\alpha F}{\partial |y|^\alpha} \tag{404}$$

with

$$\beta = 1/\ln \lambda_T, \quad \alpha = 1/\ln \lambda_y. \tag{405}$$

This result can also be applied to the GLG of squares or rotated slits (see Fig. 42, parts (b,c)) since the scatterers do not change λ_T and λ_ℓ.

Another interesting case appears for the rhombic shape of the scatterer (Fig. 40) and the corresponding GLG of rhombi (Zaslavsky and Edelman, 2002). Fig. 15 shows a sample of a trajectory and how it fills phase space. In spite of a long time ($> 10^7$) the process of covering the phase space (ℓ, v_ℓ) is so slow that ergodicity should be excluded for any, even astronomical, time. The phase space dynamics can be considered as quasi one-dimensional and formulas (299)–(302) can be applied with γ_1, i.e. density distribution of the recurrences should follow the law

$$P_{\text{rec}}(t) \sim \text{const.}/t^{2+\delta} \tag{406}$$

with a negligible value of $\delta > 0$. The simulations confirm the result with fairly high accuracy for rhombi with any irrational tangent of their angles.

Similar to the rhomb billiard case can appear in the rays propagation in a waveguide with non-uniform section (Fig. 43). This type of problem is typical for the light propagation in fiber optics.

Our last comment to this section is that sometimes the Lyapunov exponents for a problem can be so small that the pseudochaotic situation dominates and the methods of FK should be applied. This makes the necessity of the previous analysis more ubiquitous than it can be assumed from the beginning.

15.3. Maps with discontinuities

Consider a map

$$p_{n+1} = p_n + Kf(x_n), \quad x_{n+1} = x_n + p_{n+1} \tag{407}$$

which for $f(x) = \sin x$ coincides with the standard map. The map with $f(x) = \lfloor x \rfloor$ is called sawtooth map. It has zero Lyapunov exponent for $-4 < K < 0$. Nevertheless in the domain of K a pseudochaos

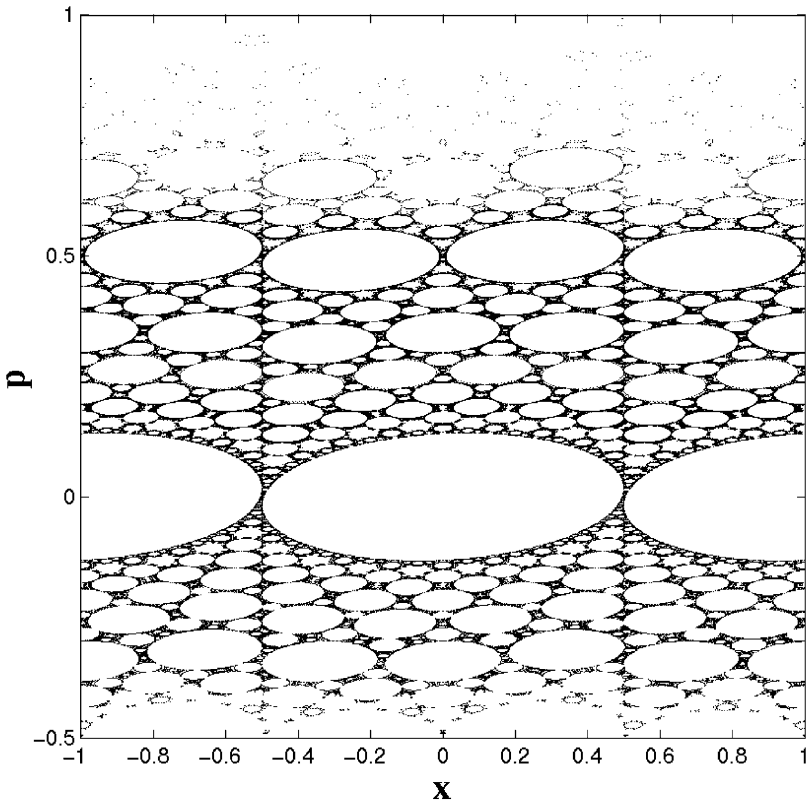

Fig. 44. Example of pseudochaotic trajectory for the standard map ($K = -0.0701\ldots$).

exists and the kinetics is fractional with the superdiffusion along x (Fig. 44). Similar situation is for the map

$$u_{n+1} = v_n, \quad v_{n+1} = -u_n - Kf(v_n) \tag{408}$$

with the same $f(x) = \lfloor x \rfloor$ instead of $\sin x$ (Fig. 45). The fractional properties of kinetics in such models are not studied yet.

16. Other applications of fractional kinetics

16.1. Fractional path integral

As a normal diffusion equation, a solution to the FKE with $\beta = 1$, $\alpha < 2$ also can be presented in a form of path integrals which we present below. Following Laskin (2000a, b), we start from the fractal generalization of the Schrödinger equation (see also West, 2000; Lutz, 2001).

Consider first the Feynman–Kac integral

$$K_{\mathrm{FK}}(x_b, t_b | x_a, t_a) = \int_{x_a}^{x_b} \hat{\mathscr{D}}_{\mathrm{FK}} x(\tau) \cdot \exp\left\{-c \int_{t_a}^{t_b} \mathrm{d}\tau\, V(\tau)\right\}, \tag{409}$$

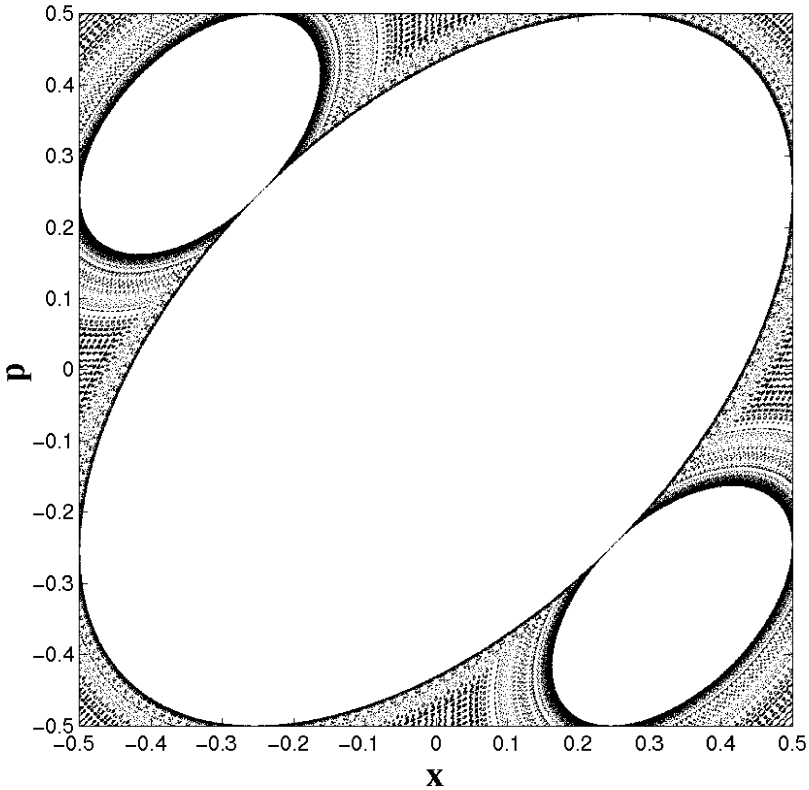

Fig. 45. Example of pseudochaotic trajectory for the web map ($K = -1.019\ldots$).

where the path is defined as a function $x(\tau)$, $x_b = x(\tau = t_B)$, $x_a = x(\tau = t_a)$, and c is a constant that will be defined later, and the functional measure $\hat{\mathscr{D}}_{\mathrm{FK}}x(\tau)$ is

$$\hat{\mathscr{D}}_{\mathrm{FK}}x(\tau) = \lim_{N\to\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N \left|\frac{2\pi\epsilon}{c}\right|^{-N/2} \cdot \exp\left\{\frac{c}{2\epsilon}(x_j - x_{j-1})^2\right\} \tag{410}$$

with

$$\epsilon = (t_b - t_a)/N \equiv \Delta t \ . \tag{411}$$

The limit $N \to \infty$ and definition (411) imply

$$\Delta x \sim x_{j+1} - x_j \sim (\Delta t/c)^{1/2}. \tag{412}$$

For free dynamics ($V \equiv 0$) and

$$K_{\mathrm{FK}}(x_b, t_B | x_a, t_a) = \left|\frac{c}{2\pi(t_b - t_a)}\right|^{1/2} \cdot \exp\left\{c\,\frac{(x_b - x_a)^2}{2(t_b - t_a)}\right\} \ . \tag{413}$$

The "wave function" is defined as

$$\psi(x, t + \epsilon) = \int_{-\infty}^{\infty} \mathrm{d}y\, K_{\mathrm{FK}}(x, t + \epsilon | y, t)\psi(y, t) \ , \tag{414}$$

where we considered the limit $t_a \to -\infty$, $t_b \to \infty$. It gives a differential equation

$$\frac{\partial\psi(x,t)}{\partial t} = \frac{1}{2c}\frac{\partial^2\psi}{\partial x^2} + V\psi \tag{415}$$

(see for example Feynman and Hibbs (1965)). For $c = \mathrm{i}/\hbar$ (415) represents the Schrödinger equation, and for $c = 1/\mathscr{D}$ it is a diffusion type equation.

Laskin (2000a, b) considered the following replacement of (410)

$$\hat{\mathscr{D}}_L x(\tau) = \lim_{N\to\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N \left|\frac{2\pi\epsilon}{c}\right|^{-N/\alpha} \cdot \prod_{j=1}^{N} L_\alpha\left(\frac{|x_j - x_{j-1}|}{(c\epsilon)^{1/\alpha}}\right) , \tag{416}$$

where $L_\alpha(x,\epsilon)$ is Lévy distribution function defined by

$$L_\alpha(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx - ct|k|^\alpha}, \quad (0 < \alpha \leqslant 2) . \tag{417}$$

Instead of (412) we have

$$\Delta x \sim (\Delta t/c)^{1/\alpha} \quad (1 < \alpha \leqslant 2) \tag{418}$$

that characterize the fractal properties of paths with a fractal dimension α.

The free path integral (413) is replaced by

$$K_{L_\alpha}(x_b, t_b | x_a, t_a) = \left|\frac{c^{\alpha-1}}{2\pi(t_b - t_a)}\right|^{1/\alpha} L_\alpha(x_b - x_a, c^{(1-\alpha)/\alpha}(t_b - t_a)^{1/\alpha}) . \tag{419}$$

Using the definition similar to (414) of the "wave function" $\psi(x,t)$ with the replacement $K_{\mathrm{FK}} \to K_{L_\alpha}$, one can get the FKE

$$\frac{\partial\psi}{\partial t} = \frac{1}{2}\mathscr{D}\frac{\partial^\alpha\psi}{\partial|x|^\alpha} + V\psi \quad (1 < \alpha \leqslant 2) \tag{420}$$

for $c = 1/\mathscr{D}$, or the fractional Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\hbar^\alpha\frac{\partial^\alpha\psi}{\partial|x|^\alpha} + V\psi \quad (1 < \alpha \leqslant 2) \tag{421}$$

for $c = \mathrm{i}/\hbar$.

16.2. Quantum Lévy flights

Quantum chaos is a manifestation of classical chaos in quantum systems (see for example Zaslavsky, 1981; Haake, 2001). Typically, classical randomness leads to a localization of quantum wave function with some localization length ℓ_q. This phenomenon leads to a saturation of the diffusion. There are strong transformations of the localization process in the cases when the classical dynamics exhibits anomalous diffusion and Lévy flights. Particularly, these special cases lead to the strong increase of the diffusion time and immense increase of the localization length ℓ_q (Roncaglia et al., 1995; Bonci et al., 1996; Stefancich et al., 1998; Sundaram and Zaslavsky, 1999; Iomin and Zaslavsky, 1999, 2000, 2001; Artuso and Rusconi, 2001). Quantum Lévy flights were discussed with respect to the density matrix evolution (Kusnezov, 1994; Kusnezov and Lewenkopf, 1996; Kusnezov et al., 1999) and for the standard map (Iomin and Zaslavsky, 2002). All these directions are at the beginning of the investigation and are out of the goal of this review.

17. Conclusion

As far as our experimental technology advances in the precision of instruments, the deeper and more important is the role of the dynamics without a noise. New kinds of physics arrives to the working bench where the unpredictable noise is replaced by the chaotic randomness which is very different from the chaos of the roulette. There is no Gaussian distribution of fluctuations and there is no diffusional type of kinetics. At the same time, there is a kind of universality of the kinetics and there is a kind of thermodynamic equilibrium. Understanding of the randomness induced by the dynamic equation is a challenging problem and its solution needs a new and a deep view of the phase space topology of the dynamics. We hope that this review, designated to bring together the phase space structure and kinetics, is one of the first steps in elucidating the problems along that way and in exposing the successful results as well as difficulties.

Acknowledgements

This article has appeared after collaboration and numerous discussions with V. Afraimovich, S. Benkadda, B. Carreras, M. Courbage, M. Edelman, A. Iomin, S. Kassibrakis, L. Kuznetsov, X. Leoncini, J. Lowenstein, O. Lyubomudrov, V. Melnikov, B. Niyazov, A. Provenzale, V. Rom-Kedar, A. Saichev, M. Shlesinger, B. Sundaram, A. Verga, A. Vulpiani, H. Weitzner, R. White, and L.-S. Young. M. Shlesinger and M. Gitterman have read the manuscript and their numerous comments have improved the text. M. Edelman helped to prepare the figures, and M. Zaslavsky created a color plate from the simulation data. Their help was invaluable.

The work was supported by the US Navy Grants No. N00014-96-1-00557 and No. N00014-97-1-0426, and by the US Department of Energy Grant No. DE-FG02-92ER54184. The simulations were performed in the supercomputer centers NERSC and NPACI at San Diego.

Appendix A. Fractional integro-differentiation

The basic introduction to the fractional calculus can be found in Gelfand and Shilov (1964), Samko et al. (1987), Miller and Ross (1993), Hilfer (1993) and Butzer and Westphal (2000).

Fractional integration of order β is defined by the operators

$$I^{\beta}_{a+}f(t) = \frac{1}{\Gamma(\beta)}\int_a^t f(\tau)(t-\tau)^{\beta-1}\,\mathrm{d}\tau \quad (\beta > 0)\ ,$$

$$I^{\beta}_{b-}f(t) = \frac{1}{\Gamma(\beta)}\int_t^b f(\tau)(\tau-t)^{\beta-1} \quad (\beta > 0)\ . \tag{A.1}$$

There is no constraint on the limits (a, b) and one can use different types of definitions (see the encyclopedic book by Samko et al., 1987). Any of them can be used for applications, and choice

of a specific definition is only a matter of convenience. We put everywhere $a=-\infty$, $b=\infty$ and simplify notations (A.1):

$$I_\beta f(t) \equiv \frac{1}{\Gamma(\beta)} \int_{-\infty}^{t} f(\tau)(t-\tau)^{\beta-1}\,\mathrm{d}\tau \quad (\beta>0)\ ,$$

$$I_\beta f(-t) \equiv \frac{1}{\Gamma(\beta)} \int_{t}^{\infty} f(\tau)(\tau-t)^{\beta-1}\,\mathrm{d}\tau \quad (\beta>0)\ . \tag{A.2}$$

Definitions (A.2) can be written in a compact form:

$$I_\beta f(t) = f(t) \bigstar \frac{t_+^{\beta-1}}{\Gamma(\beta)}\ ,$$

$$I_\beta f(-t) = f(t) \bigstar \frac{t_-^{\beta-1}}{\Gamma(\beta)}\ , \tag{A.3}$$

where $\bigstar$ means convolution and

$$t_+^{\beta-1} = \begin{cases} t_+^{\beta-1}, & t>0\ , \\ 0, & t<0\ , \end{cases}$$

$$t_-^{\beta-1} = \begin{cases} t_-^{\beta-1}, & t<0\ , \\ 0, & t>0\ . \end{cases} \tag{A.4}$$

The fractional derivative could be defined as an inverse operator to I_β, i.e.

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} = I_{-\beta}, \quad I_\beta = \frac{\mathrm{d}^{-\beta}}{\mathrm{d}t^{-\beta}}\ . \tag{A.5}$$

Their explicit form is

$$\frac{\mathrm{d}^\beta f(t)}{\mathrm{d}t^\beta} = \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} \frac{f(\tau)\,\mathrm{d}\tau}{(t-\tau)^\beta}\ ,$$

$$\frac{\mathrm{d}^\beta f(t)}{\mathrm{d}(-t)^\beta} = \frac{-1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t}^{\infty} \frac{f(\tau)\,\mathrm{d}\tau}{(\tau-t)^\beta} \quad (0<\beta<1)\ . \tag{A.6}$$

Expressions (A.1), (A.2) and (A.6) are called Riemann–Liouville integrals and derivatives, respectively.

For arbitrary β we have the definition

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\infty}^{t} \frac{f(\tau)\,\mathrm{d}\tau}{(t-\tau)^{\beta-n+1}}\ ,$$

$$\frac{\mathrm{d}^\beta}{\mathrm{d}(-t)^\beta} f(t) = \frac{-1}{\Gamma(n-\beta)} \frac{\mathrm{d}^n}{\mathrm{d}(-t)^n} \int_{t}^{\infty} \frac{f(\tau)\,\mathrm{d}\tau}{(\tau-t)^{\beta-n+1}}\ , \tag{A.7}$$

where $n=[\beta]$ is an integer part of $\beta>0$.

An important application is the Riesz derivative:

$$(-\Delta)^{\alpha/2} \equiv \frac{\mathrm{d}}{\mathrm{d}|x|^\alpha} = -\frac{1}{2\cos(\pi\alpha/2)}\left[\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} + \frac{\mathrm{d}^\alpha}{\mathrm{d}(-x)^\alpha}\right] \quad (\alpha \neq 1) \tag{A.8}$$

which in a regularized form reads

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}|x|^\alpha} f(x) = -\frac{1}{K(\alpha)} \int_{0+}^{\infty} \frac{\mathrm{d}y}{y^{\alpha+1}} [f(x-y) - 2f(x) + f(x+y)], \quad 0 < \alpha < 2 \ ,$$

$$K(\alpha) = \begin{cases} 2\Gamma(-\alpha)\cos(\pi\alpha/2), & \alpha \neq 1 \ , \\ -\pi, & \alpha = 1 \ . \end{cases} \tag{A.9}$$

Appendix B. Useful formulas of fractional calculus

Some simple formulas help to perform calculations. For a few differentiations we have

$$\frac{\mathrm{d}^{\alpha+\beta}}{\mathrm{d}t^{\alpha+\beta}} = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} = \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \tag{B.1}$$

For a power function

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha} \ . \tag{B.2}$$

Particularly

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t > 0 \tag{B.3}$$

and

$$\lim_{\alpha\to 1} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} 1 = \delta(t) \ . \tag{B.4}$$

Let us define the Fourier transform as

$$g(q) = \int_{-\infty}^{\infty} g(x)\mathrm{e}^{\mathrm{i}qx}\,\mathrm{d}x \tag{B.5}$$

with a notation

$$g(x) \xrightarrow{F} g(q) \ . \tag{B.6}$$

Then

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} g(x) \xrightarrow{F} (-\mathrm{i}q)^\alpha g(q) \ ,$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}(-x)^\alpha} g(x) \xrightarrow{F} (\mathrm{i}q)^\alpha g(q) \tag{B.7}$$

and for the Riesz derivative

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}|x|^\alpha} g(x) \xrightarrow{F} -|q|^\alpha g(q) \ . \tag{B.8}$$

Let us denote

$$[g(x)\cdot f(x)] = \int_{-\infty}^{\infty} g(x) f(x)\,\mathrm{d}x \tag{B.9}$$

as the scalar product of $g(x)$ and $f(x)$. The following formula is equivalent to integration by parts:

$$\left[g(x)\cdot \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} f(x)\right] = \left[f(x)\cdot \frac{\mathrm{d}^\alpha}{\mathrm{d}(-x)^\alpha} g(x)\right]. \tag{B.10}$$

It can be used to prove that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} f(x) = \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{d}^\alpha}{\mathrm{d}(-x)^\alpha} f(x) = \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\mathrm{d}^\alpha}{\mathrm{d}|x|^\alpha} f(x) \equiv 0 \tag{B.11}$$

if $f(q=0)\neq 0$.

In addition to (A.8) and (B.8) let us define Riesz derivative for the m-dimensional case, i.e. $x\in R^m$ (Samko et al., 1987). Let us define a finite difference along $y\in R^m$ as

$$\Delta_y^\ell f(x) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} f(x-ky)\ . \tag{B.12}$$

Then

$$(-\Delta)^{\alpha/2} f(x) = \frac{1}{d_{m,\ell}(\alpha)} \int_{R^m} \frac{\Delta_y^\ell f(x)}{|y|^{m+\alpha}}\,\mathrm{d}y \tag{B.13}$$

with $\ell > \alpha$ and

$$d_{m,\ell}(\alpha) = \int_{R^m} (1-\mathrm{e}^{-\mathrm{i}z})^\ell |y|^{-m-\alpha}\,\mathrm{d}y\ . \tag{B.14}$$

The Fourier transform gives

$$(-\Delta)^{\alpha/2} \xrightarrow{F} |k|^\alpha, \quad k\in R^m \tag{B.15}$$

which has the correct sign for $\alpha = 2$.

Appendix C. Solutions to FKE

There are different possibilities to express the solutions to FKE through special functions depending on the type of the equation and on the boundary condition (see for example Wyss, 1986; Schneider and Wyss, 1989; West et al., 1997; Metzler and Klafter, 2000). One of the possibilities is to use the Fox functions (Fox, 1961; West et al., 1997). We present below a simplified version of the solution what is expressed through the Mittag–Lefler function (see more in Saichev and Woyczynski, 1997):

$$E_\beta(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m\beta+1)}\ . \tag{C.1}$$

For $\beta = 1$:

$$E_1(z) = \mathrm{e}^z$$

and for $\beta = 1/2,\ z = -s \leqslant 0$:

$$E_{1/2}(-s) = \frac{2}{\sqrt{\pi}}\,\mathrm{e}^{s^2}\int_x^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y \tag{C.2}$$

For FKE in the form

$$\frac{\partial^\beta}{\partial t^\beta}P(q,t) + |q|^\alpha P(q,t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t > 0 \tag{C.3}$$

and

$$P(q,t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}qx}P(x,t)\,\mathrm{d}x \tag{C.4}$$

we have simply

$$P(q,t) = E_\beta(-|q|^\alpha t^\beta) \tag{C.5}$$

or

$$P(x,t) = \frac{1}{\pi|y|}\,\frac{1}{t^\mu}\sum_{m=0}^{\infty}\frac{(-1)^m}{|y|^{m\alpha}}\,\frac{\Gamma(m\alpha+1)}{\Gamma(m\beta+1)}\cdot\cos\left[\frac{\pi}{2}(m\alpha+1)\right], \tag{C.6}$$

with

$$y = x/t^{\mu/2}, \quad \mu = 2\beta/\alpha\ . \tag{C.7}$$

The asymptotic solution from (C.6) is

$$P(x,t) \sim \frac{1}{\pi}\,\frac{t^\beta}{|x|^{\alpha+1}}\,\frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\sin\frac{\pi\alpha}{2}\ , \tag{C.8}$$

where

$$x^2 \gg t^\mu = t^{2\beta/\alpha}\ . \tag{C.9}$$

Particularly the moments

$$\langle |x|^\delta\rangle = \int_{-\infty}^{\infty}|x|^\delta P(x,t)\,\mathrm{d}x \tag{C.10}$$

are bounded if $0 < \delta < \alpha < 2$ as it follows from (C.8).

Another possibility to use integral representation of the solution in (p,k)-space and evaluating of its asymptotics was considered in Weitzner and Zaslavsky (2001).

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